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Toeplitz Operators on Flows*

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Let $\mathbb{R}$ act continuously on a compact Hausdorff space X giving rise to a flow on X, let $\varphi \in C(X)$, and let T_{φ_x} denote the Toeplitz operator on $H^2(\mathbb{R})$ determined by the function φ_x on $\mathbb{R}$ defined by $\varphi_x(t) = \varphi(x+t)$. In this paper, we investigate the relation between the spectral properties of T_{φ_x}, the dynamical properties of the flow, and the value distribution theory of φ. The analysis proceeds by imbedding T_{φ_x} in a type II_∞ factor and computing the real-valued index of the operator à la Connes. Our sharpest invertibility result asserts that if the flow is strictly ergodic and if the asymptotic cycle determined by the flow is injective on $H^1(X, \mathbb{Z})$, then T_{φ_x} is invertible if and only if φ does not vanish on X and determines the zero element in $H^1(X, \mathbb{Z})$. This generalizes the classical result of Gohberg and Krein and its extension to Toeplitz operators with almost periodic symbols due to Coburn, Douglas, Schaeffer, and Singer. When φ is analytic, in the sense that φ_x belongs to $H^\infty(\mathbb{R})$ for all x, we relate the II_∞ index of the Toeplitz operator determined by φ with the density of the zeros of φ_x in the upper half-plane. Much of our efforts to achieve this result are devoted to generalizing to arbitrary flows the value distribution theory of analytic almost periodic functions developed by Bohr, Jessen, and Tornehave, and others. © 1990 Academic Press, Inc.

INTRODUCTION

1.1. In recent years the philosophy has emerged that if one wants to study operators on Hilbert space that exhibit certain types of random behavior, then one may be able to embed the operators into a II_∞ factor and to use the relative trace on the factor to calculate properties of the

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operator. Perhaps the most spectacularly successful application of this philosophy is Connes's index theory for foliations [Co1, Co2]. An important forerunner of this result was the work of Coburn, Douglas, Schaeffer, and Singer [CDSS] devoted to the spectral properties of Toeplitz operators with almost periodic symbols. What they showed is that while such an operator is never Fredholm in the ordinary sense unless it is invertible, the operator can be embedded in a II_∞ factor with the result that the embedded operator is Fredholm in the generalized sense of Breuer if and only if the symbol of the operator is invertible. In that case, the Breuer–Fredholm index of the operator is calculated to be the negative of the mean motion of the symbol.

The present paper is the result of the confluence of two streams of thought. First, we wanted to see how Connes's theory could be used to study Toeplitz operators on the line whose symbols are continuous but which oscillate more wildly at infinity than almost periodic functions. It turns out that generalized index theorems and invertibility criteria can be found but, strictly speaking, they are not corollaries of Connes's work. The philosophy underlying his work and ours is the same, but interesting differences and problems emerge. Secondly, we wanted to see how the spectral theory of Toeplitz operators could be used to gain insight into the function theoretic properties of certain types of random analytic functions. On the disk, the ordinary Fredholm index of a Fredholm–Toeplitz operator with continuous analytic symbol is (the negative of) the number of zeros of the symbol in the disc. In our theory, it turns out that under suitable hypotheses, the generalized index of a generalized Fredholm–Toeplitz operator with analytic symbol on a flow is (the negative of) the density of the zeros of the function calculated in any half-plane erected over any orbit of the flow. The connection between indices and value distribution theory is made through the Pincus principal function [CaP1] and through a careful reworking and modernization of parts of the classical value distribution theory for analytic almost periodic function, particularly the work of Jessen and Tornehave [JT]. As a result, we will uncover some surprising connections between the topology of a flow and the value distribution theory of analytic functions defined on it.

We turn now to a somewhat more detailed introduction of our results.

1.2. Concerning von Neumann algebras, we follow the notation and terminology of Dixmier [Dix]. Let $\mathfrak{M}$ be a semifinite von Neumann algebra and let τ be a faithful, normal, semifinite trace on $\mathfrak{M}$. (Since we consider not other kind of traces, we drop the adjectives "faithful," "normal," and "semifinite.") A projection $E \in \mathfrak{M}$ is called *finite, or relatively finite dimensional*, if $\tau(E) < \infty$, and $\tau(E)$ is called the (relative) *dimension* of E. An operator $T \in \mathfrak{M}$ is said to have (relatively) *finite rank* if there is a finite projection $E \in \mathfrak{M}$ such that $ETE = T$. The collection $\mathfrak{K}_0(\mathfrak{M})$ of all finite rank operators is a two-sided ideal in $\mathfrak{M}$ whose norm closure, $\mathfrak{K}_\infty(\mathfrak{M})$, is called the ideal of (relatively) *compact* operators in $\mathfrak{M}$. The functional $T \to \tau((T^*T)^{p/2})$, $T \in \mathfrak{K}_0(\mathfrak{M})$, is a norm on $\mathfrak{K}_0(\mathfrak{M})$ for each p, $1 \leqslant p < \infty$, and the completion of $\mathfrak{K}_0(\mathfrak{M})$ in this norm is denoted $L^p(\mathfrak{M}, \tau)$. The elements in $L^p(\mathfrak{M}, \tau)$ can be realized as operators (possibly unbounded) that are affiliated with $\mathfrak{M}$. We denote by $\mathfrak{K}_p(\mathfrak{M})$, $1 \leqslant p < \infty$, the intersection $\mathfrak{M} \cap L^p(\mathfrak{M}, \tau)$. $\mathfrak{K}_1(\mathfrak{M})$ is the ideal of (relative) *trace class operators* and $\mathfrak{K}_2(\mathfrak{M})$ is the ideal of (relative) *Hilbert-Schmidt operators*. The latter is an achieved Hilbert algebra under the inner product $(A, B) = \tau(B^*A)$, $A, B \in \mathfrak{M}$.

An operator $T \in \mathfrak{M}$ is called *relatively Fredholm*, or, as we shall say, T is *Breuer–Fredholm*, if the image of T in $\mathfrak{M}/\mathfrak{K}_\infty(\mathfrak{M})$ is invertible. As is shown in [Br], T is Beuer–Fredholm if and only if the ranges of T and T^* contain subspaces affiliated to $\mathfrak{M}$ that are co-finite dimensional; i.e., if and only if there are projections E and E_* in $\mathfrak{M}$ such that the range of T contains the range of E, while the range of T^* contains the range of E_*, and such that $I - E$ and $I - E_*$ lie in $\mathfrak{K}_0(\mathfrak{M})$. If T is Breuer–Fredholm, the null spaces of T and T_* are (relatively) finite dimensional and the Breuer–Fredholm *index* of T, Index T, is defined to be $\tau(N(T)) - \tau(N(T^*))$, where $N(T)$ and $N(T^*)$ denote the projections onto the null spaces of T and T^*. The Breuer–Fredholm theory parallels the classical Fredholm theory quite closely, but there is an important difference: In the Breuer–Fredholm theory, Breuer–Fredholm operators need not have closed range. This causes us some problems.

1.3. Throughout this paper, X will denote a compact Hausdorff space on which the real line $\mathbb{R}$ acts continuously as a transformation group. We denote by $x + t$ the translate of an x in X by a t in $\mathbb{R}$. We refer to $(X, \mathbb{R})$ as a *flow*. It is called *minimal* if there are no nontrivial closed invariant sets, and it is called *strictly ergodic* if it is minimal and if there is exactly one invariant probability measure on X. The Kakutani–Markov fixed point theorem guarantees the existence of at least one invariant probability measure on a flow, but the assumption that the measure is unique is special. However, many flows are strictly ergodic and there is even a sense in which the generic flow is strictly ergodic [DE].

Recall that in general, the first Čech cohomology group of X with integer coefficients, $H^1(X, \mathbb{Z})$, is isomorphic to the quotient group $C(X)^{-1}/\exp(C(X))$, where $C(X)^{-1}$ denotes the group (under pointwise multiplication) of invertible elements in $C(X)$ and $\exp(C(X))$ denotes the subgroup of functions in $C(X)$ of the form $\exp(\psi)$, $\psi \in C(X)$. Each coset in $C(X)^{-1}/\exp(C(X))$ contains and element φ that is differentiable along orbits; i.e., if $(D_h\varphi)(x) = (\varphi(x+h) - \varphi(x))/h$, then $\lim_{h \to 0} D_h\varphi$ exists in

$C(X)$. The limit is denoted φ'. If $[\varphi]$ denotes the coset of φ in $C(X)^{-1}/\exp(C(X))$, then the *topological index* of φ or of $[\varphi]$, $\mu([\varphi]; m)$, determined by $[\varphi]$ and an invariant probability measure m on X is defined by the formula

$$\mu(\varphi; m) = \mu([\varphi]; m) = \frac{1}{2\pi i} \int_X \frac{\varphi_0'}{\varphi_0}\, dm,$$

where $[\varphi_0] = [\varphi]$, and φ_0 is differentiable. The topological index map $\mu(\cdot\,; m)$ is a homomorphism of $H^1(X, \mathbb{R})$ into $\mathbb{R}$. (These facts are due to Schwartzman [Schw] and will be discussed in greater detail in Section 4.)

1.4. Let $H^2(\mathbb{R})$ denote the usual Hardy space of the upper half-plane viewed as a subspace of $L^2(\mathbb{R})$. For $\varphi \in L^\infty(\mathbb{R})$, the *Toeplitz operator* on $H^2(\mathbb{R})$ with *symbol* φ, T_φ, is defined by the formula

$$T_\varphi \xi = P(\varphi \xi),$$

where $\xi \in H^2(\mathbb{R})$ and P denotes the orthogonal projection of $L^2(\mathbb{R})$ onto $H^2(\mathbb{R})$. Observe that if $(X, \mathbb{R})$ is a flow, then for each $\varphi \in C(X)$ and each $x \in X$, we obtain a Toeplitz operator with sqymbol φ_x, where $\varphi_x(t) = \varphi(x+t)$. We write T_φ^x instead of T_{φ_x}, and we write $\mathfrak{T}_x$ for the C^*-algebra generated by $\{T_\varphi^x \mid \varphi \in C(X)\}$. We are interested in learning how the topological properties of X are reflected in the spectral properties of the T_φ^x's and in the algebraic properties of $\mathfrak{T}_x$.

It should be noted that in a sense we are proposing a scheme for studying the most general Toeplitz operator whose symbol is a bounded, uniformly continuous function on $\mathbb{R}$. Indeed, if φ_0 is a such a function, and if X is the maximal ideal space of the smallest translation invariant C^*-subalgebra of $L^\infty(\mathbb{R})$ containing φ_0 (we call X the *hull* of φ_0), then $\mathbb{R}$ acts continuously on X and $\varphi_0(t) = \varphi(x_0 + t)$, where x_0 is a certain point in X and φ is the Gelfand transform of φ_0. Thus, $T_{\varphi_0} = T_\varphi^{x_0}$. The problem, of course, is that it is difficult to determine the properties of X from φ_0. So, rather than viewing our efforts as an approach to uncovering the spectral properties of a given Toeplitz operator with bounded uniformly continuous symbol, they should be thought of as helping to demonstrate and to organize the complexity that these operators may exhibit.

Let m be an invariant, ergodic, probability measure on X and let $L^2(X \times \mathbb{R})$ be the L^2-space based on the product of m with Lebesgue measure on $\mathbb{R}$. For $\varphi \in L^\infty(X)$, we write $\sigma^m(\varphi)$ for the operator on $L^2(X, \mathbb{R})$ defined by the formula $\sigma^m(\varphi)\, \xi(x, s) = \varphi(x)\, \xi(x, s)$ and for $t \in \mathbb{R}$, we write U_t^m for the unitary operator on $L^2(X \times \mathbb{R})$ defined by $(U_t^m \xi)(x, s) = \xi(x+t, s-t)$. Then, assuming m is not supported on a periodic orbit, the von Neumann algebra generated by $\{\sigma^m(\varphi) \mid \varphi \in L^\infty(X)\}$ and $\{U_t^m\}_{t \in \mathbb{R}}$ is a II_∞-factor, denoted $L^\infty(X) \rtimes \mathbb{R}$, and is called the *group-measure algebra* determined by $(X, \mathbb{R})$ and m. Let P^m be the spectral projection of $\{U_t^m\}_{t \in \mathbb{R}}$ corresponding to $[0, \infty)$, let $H^2(X \times \mathbb{R})$ be the range of P^m, and let $\mathfrak{N} = P^m(L^\infty(X) \rtimes \mathbb{R})\, P^m$. Then $\mathfrak{N}$ is also a II_∞-factor. For $\varphi \in C(X)$, we define the Toeplitz operator T_φ^m in $\mathfrak{N}$ be the formula

$$T_\varphi^m \xi = P^m \sigma^m(\varphi)\, \xi,$$

$\xi \in H^2(X \times \mathbb{R})$, and we write $\mathfrak{T}_m$ for the C^*-algebra generated by $\{T_\varphi^m \mid \varphi \in C(X)\}$.

1.5. With these preliminaries in hand, we may now state our basic spectral theorem as follows. The proof combines Theorems 19, 24.4, 25.2, and 26.1, below.

THEOREM. *Suppose that $(X, \mathbb{R})$ is minimal and that m is an invariant ergodic probability on X. Then*

(i) *The map $T_\varphi^x \to T_\varphi^m$, $\varphi \in C(X)$, extends to a C^*-isomorphism from $\mathfrak{T}_x$ onto $\mathfrak{T}_m$ for each $x \in X$. In particular, all the $\mathfrak{T}_x$'s are isomorphic.*

(ii) *For $\varphi \in C(X)$, T_φ^m is a Breuer–Fredholm operator in $\mathfrak{N}$ if and only if $\varphi \in C(X)^{-1}$. In this case, the Breuer–Fredholm index of T_φ^m,* Index $\mathfrak{T}_\varphi^m$, *is $-\mu(\varphi; m)$.*

(iii) *If $(X, \mathbb{R})$ is strictly ergodic, so that m is the unique invariant probability measure on X, and if $\mu(\cdot\,; m)$ is an injective homomorphism of $H^1(X, \mathbb{Z})$, then for $\varphi \in C(X)$ and $x \in X$, T_φ^x (or T_φ^m) is invertible if and only if $\varphi \in C(X)^{-1}$ and $\mu(\varphi; m) = 0$.*

1.6. We give two proofs of the index formula of Theorem 1.5 in Theorem 25.2. One is more or less direct, and applies as well to systems of Toeplitz operators (these are treated in Section 32). It is similar to an argument of Schaeffer [Sch]. The other is based on ideas of Carey and Pincus. Suppose that $(X, \mathbb{R})$ and m satisfy the blanket assumptions of Theorem 1.5, and suppose that φ and ψ are differentiable functions on X. We show in Theorem 23.1 that then the commutator $[T_\varphi^m, T_\psi^m]$ lies in $\mathfrak{K}_1(\mathfrak{N})$ and its relative trace is $-(1/2\pi i) \int_X \varphi'(x)\, \psi(x)\, dm(x)$. As a result, we are able to compute the Pincus principal function, $g(\varphi; z)$, of T_φ^m, for φ differentiable, and find in Theorem 25.1 that it is

$$-\frac{1}{2\pi i} \int_X \frac{\varphi'(x)}{\varphi(x) - z}\, dm(x).$$

This formula was established for special periodic functions in [CaP2, p. 492] and the point stressed in [CaP1, CaP2] is that $g(\varphi; z)$ carries indicial information about T_φ^m even for z in the essential spectrum of T_φ^m. This view becomes particularly clear when φ is *analytic* on X in the sense that for each $x \in X$, the function of t, $\varphi(x+t)$, extends to be a bounded analytic function in the upper half-plane. We denote the collection of all such functions by $A(X, \mathbb{R})$.

To understand how $g(\varphi; \cdot)$ gives indicial information about T_φ^m for $\varphi \in A(X, \mathbb{R})$, we need a bit more notation. Given $\varphi \in A(X, \mathbb{R})$ and $y > 0$, let $\varphi_y(x) = (1/\pi) \int_X \varphi(x+t)\, y/(y^2+t^2)\, dt$. Then, evidently, φ_y is differentiable along orbits and so we can form $g(\varphi_y; \cdot)$. On the other hand, for x and y fixed, $\varphi_y(x+t)$, as a function of t, is the restriction of a bounded analytic function to the line $z = iy$. Consequently, we may attempt to define its *mean motion* $\mu(\varphi_y; x)$ in a generalized sense appropriate to analytic functions. (This notion is not new with us; it has its roots in the value distribution theory of analytic almost periodic functions. We will reproduce what we need in Section 8.) Also, a theorem of Carl Carlson [Car] implies that $\log|\varphi_y|$ is integrable with respect to m (see Theorem 11.1 below). We set $\Phi(\varphi; y) = (1/2\pi) \int \log|\varphi_y|\, dm$, and following Jessen and Tornehave [JT] we call $\Phi(\varphi; \cdot)$ the Jensen function of φ. As the name implies, $\Phi(\varphi; \cdot)$ carries information about the zeros of the analytic functions obtained by extending φ to the upper half-planes erected above the orbits of the flow. In particular, as we shall show in Theorem 11.4, $\Phi(\varphi; \cdot)$ is convex, so that the derivative of $\Phi(\varphi; y)$ with respect to y, $\Phi'(\varphi; y)$, exists for all but countably many y. Theorem 28.1, which generalizes a result of Carey and Pincus in [CaP2], asserts that if the flow is strictly ergodic, then for each $y > 0$, there is a planar null set E_y such that for $z \notin E_y$ the mean motion $\mu(\varphi_y - z; x)$ exists for every $x \in X$, $\Phi(\varphi - z; y)$ is differentiable in y, and

$$-g(\varphi_y; z) = \mu(\varphi_y - z; x) = \frac{\partial}{\partial y} \Phi(\varphi - z; y).$$

The point is as follows: If z is not in the range of φ_y so that $\varphi_y - z$ is invertible, then $-\mu(\varphi_y - z; x)$ would be $\operatorname{Index}(T_{\varphi-z}^m)$. However, for some z's not in E_y, $\varphi_y - z$ may not be invertible, yet $g(\varphi_y; z)$ makes sense and may be viewed as an average winding number, i.e., an index.

1.7. The paper is divided into two parts. In Part I we generalize to the context of functions analytic on flows some of the basic facts about the value distribution theory of analytic almost periodic functions. There are two key sets of results here. The first, Theorem 14.1 and its corollaries, shows that if $(X; \mathbb{R})$ is strictly ergodic and if $f \in A(X, \mathbb{R})$, then for two points y_1, y_2, $0 < y_1 < y_2 < \infty$, where the derivative of the Jensen function $\Phi(f; y)$ exists, and for any $x \in X$,

$$\Phi'(f; y_2) - \Phi'(f; y_1)$$

represents the density $H(x; y_1, y_2)$ of zeros of the analytic function $F(x; z) \equiv f_y(x+t)$, $z = t + iy$, in the strip $y_1 < \operatorname{Im} z < y_2$. That is, if $N(x; S, T, y_1, y_2)$ represents the number of zeros of $F(x; \cdot)$ in the rectangle $\{S < \operatorname{Re} z < T,\ \ y_1 < \operatorname{Im} z < y_2\}$, then the limit $\lim_{T-S-\infty}(1/(T-S))$ $N(x; S, T, y_1, y_2)$ exists and is denoted $H(x; y_1, y_2)$; it is calculated via Φ as indicated. Also, the relation between H, Φ, and the generalized mean motion μ is discussed in detail. The second key set of results, Theorem 16.1 and its corollaries, in a sense, is an integrated version of the first. We show that for each $y > 0$, there is a Lebesgue null set $E_y \subseteq \mathbb{C}$ such that for $\zeta \notin E_y$, $(\partial/\partial y)\, \Phi(f - \zeta; y)$ exists. Thus for almost all $\zeta \in \mathbb{C}$, the density of the zeros of $F - \zeta$ in any precribed strip $y_1 < \operatorname{Im} z < y_2$ can be calculated with the aid of Φ.

Part II is concerned with Toeplitz operators. In Sections 17–19, we set up the II_∞-context in which we want to study Toeplitz operators. The crucial result is Theorem 19 which is assertion (i) of Theorem 1.5. Section 20 is a digression in which we compare our analysis with Connes's index theory for foliations and raise several problems that have troubled us. In Sections 22 and 23, we evaluate the traces of certain commutators and use these in Section 25 to evaluate the Pincus principal function of T_φ^m when φ is differentiable and to prove our index theorem, Theorem 25.2, which is assertion (ii) of Theorem 1.5. Section 26 contains our invertibility criterion for Toeplitz operators on flows, Theorem 26.1; it is assertion (iii) of Theorem 1.5. In Sections 27 and 28, we derive some function theoretic consequences of our index theorem. In particular, we relate $\operatorname{Index}(T_\varphi^m)$ to the distribution of zeros of φ when φ lies in $A(X, \mathbb{R})$. Section 29 is concerned with the continuity properties of principal functions and mean motions. In Section 30, we examine consequences of the hypotheses that $H^1(X, \mathbb{Z}) = 0$. Note that if this happens, then automatically the mean motion is injective; so if $(X, \mathbb{R})$ is strictly ergodic, we conclude from Theorem 1.5 that for $\varphi \in C(X)$, and for $x \in X$, the Toeplitz operator T_φ^x is invertible if and only if φ_x is bounded away from zero. Of course it is known that for certain types of functions $\varphi \in L^\infty(\mathbb{R})$, T_φ is invertible if and only if φ is bounded away from zero. However, what is remarkable here is that there are whole translation invariant C^*-subalgebras of $L^\infty(\mathbb{R})$ consisting of functions with this property. Section 31 is concerned with the isomorphisms between algebras of Toeplitz operators and how they relate the conjugacy invariants of flows. Finally, we prove an index theorem for systems of Toeplitz operators in Section 32.

1.8. All Hilbert spaces considered will be complex and *separable*. Likewise, our compact Hausdorff spaces X will be separable. These hypotheses are not always necessary, but they are convenient and it does no material harm to assume them. We leave it to the reader to decide the limits of their necessity.

PART I

2. Our objective here is to give an account of the function theory on flows that we will need. Basically, we shall show that all of the machinery developed to study the value distribution theory for analytic almost periodic functions can be developed also for analytic functions on general flows, the most complete development coming when the flow is strictly ergodic. Throughout, $(X, \mathbb{R})$ will denote a fixed flow with X compact and separable. We will not now make the blanket assumption that $(X, \mathbb{R})$ is strictly ergodic because, when it is possible to do so, we want to state our results with the greatest possible generality for future use elsewhere. It is not clear to us yet that there is something essential about strict ergodicity in this subject. Perhaps the hypothesis arises simply as an (unwanted) artifact of our proofs. In any event, we will signal its use each time it occurs.

3.1. DEFINITION. A point $x \in X$ is called *quasi-regular* if for each function $f \in C(X)$, the limit

$$\lim_{T-S \to \infty} \frac{1}{T-S} \int_S^T f(x+t)\, dt \tag{3.1}$$

exists.

3.2. It is not hard to see (cf. [Ox]) that if x is a quasi-regular point in X, then there is an invariant probability measure m_x on X such that the limit in (3.1) is $\int f\, dm_x$. On the other hand, if m is an invariant *ergodic* probability measure on X, then by the separability of $C(X)$ and the individual ergodic theorem, there is an m-null set whose complement consists of quasi-regular points. For these x, $m_x = m$. In fact, the non-quasi-regular points form a set which is null for every invariant probability measure on X [Ox]. If $(X, \mathbb{R})$ is strictly ergodic and if m denotes the unique invariant probability measure on X, then every $x \in X$ is quasi-regular, $m_x = m$, of course, and the limit (2.1) is uniform in x (cf. [Ox]). More generally, we have the following lemma that will be useful later.

3.3. LEMMA. *Suppose that $(X, \mathbb{R})$ is strictly ergodic with unique invariant probability measure m and that $g \in C(X \times Y)$, where Y is another compact space. Then*

$$\lim_{T-S \to \infty} \frac{1}{T-S} \int_S^T g(x+t, y)\, dt = \int g(x, y)\, dm(x)$$

uniformly in x and y.

Proof. If g were a function of x alone, as just noted, the assertion is a consequence of [Ox]. However, it is clear that the set of functions for which the assertion holds is closed in $C(X \times Y)$, linear, and contains all functions of the form $g(x, y) = h(x)\, k(y)$, where $h \in C(X)$, $k \in C(Y)$. Thus, the asserted limit exists uniformly on $X \times Y$ for all g in $C(X \times Y)$.

4. Suppose that f is an invertible element of $C(X)$; i.e., suppose that $f \in C(X)^{-1}$. Then for each $x \in X$, $f(x+t)$ is a nonvanishing function on $\mathbb{R}$. We may therefore define a continuous branch of the argument of $f(x+t)$, $\arg(f(x+t))$, and for $S < T$, we may calculate the change in argument of f along the orbit of x from $x+S$ to $x+T$. We write $\Delta_{(x;S,T)} \arg f = \arg(f(x+T)) - \arg(f(x+S))$. Note that this quantity does not depend on the branch of the argument of $f(x+t)$ that is chosen. We define the *mean motion* of f along the orbit of x by the formula

$$\mu(f; x) = \lim_{T-S \to \infty} \frac{1}{2\pi(T-S)} \Delta_{(x;S,T)} \arg f, \tag{4.1}$$

provided the limit exists. It is shown in [Schw] that if x is a quasi-regular point in X, then the limit exists. As a function on the quasi-regular points of X, $\mu(f; \cdot)$ evidently is invariant and measurable. It follows that if m is an invariant ergodic probability measure on X, then for m-almost all x, $\mu(f; \cdot)$ is independent of x. We therefore denote the common value of the $\mu(f; x)$'s by $\mu(f)$, or $\mu(f; m)$ if we wish to emphasize the dependence on m. The following lemma is extracted from [Schw] and plays a basic role in the sequel. In it, as in the introduction, we identify the first Čech cohomology group of X, with integer coefficients, with $C(X)^{-1}/\exp(C(X))$. Also, we write $C^1(X)$ for the space of functions in $C(X)$ that are differentiable along orbits.

4.1. LEMMA. *For each quasi-regular point x, $\mu(\cdot\,; x)$ determines a homomorphism $\tilde{\mu}(\cdot\,; x)$ from $H^1(X, \mathbb{Z})$ into $\mathbb{R}$ by the formula*

$$\tilde{\mu}([f]; x) = \mu(f; x),$$

where $[f]$ is the coset of f in $C(X)^{-1}/\exp(C(X))$. If m is an invariant ergodic probability measure, then for m-almost all $x \in X$ we have

$$\mu(f; x) = \frac{1}{2\pi i} \int_X \frac{f'(y)}{f(y)}\, dm(y) \tag{4.2}$$

for all $f \in C^1(X)$. In particular, if $(X, \mathbb{R})$ is strictly ergodic, then $\mu(f; x)$ is constant in x and for differentiable f, $\mu(f)$ is given by Eq. (4.2), *where m is the unique invariant probability measure.*

4.2. *Remark.* We note in passing that Schwartzmann [Schw] considers only unimodular functions in his study of μ. However, that is an inessential restriction, as may easily be seen. For example, to prove (4.2), note that by the individual ergodic theorem, we have for almost all x that

$$\begin{aligned}
&-\frac{1}{2\pi i} \int_X \frac{f'(x)}{f(x)}\, dm(y) \\
&\quad = \lim_{T-S\to\infty} -\frac{1}{2\pi i} \frac{1}{T-S} \int_S^T \frac{f'(x+t)}{f(x+t)}\, dt \\
&\quad = \lim_{T-S\to\infty} \left\{ \frac{1}{(T-S)\, 2\pi i} (\log|f(x+T)| - \log|f(x+S)|) \right. \\
&\qquad \left. + \frac{1}{(T-S)\, 2\pi} (\Delta_{(x;S,T)} \arg f) \right\} \\
&\quad = \mu(f; x) + \lim_{T-S\to\infty} \frac{1}{(T-S)\, 2\pi i} (\log|f(x+T)| - \log|f(x+S)|).
\end{aligned}$$

The last limit is zero because, for each x, $|f(x+t)|$, as a function of t, is bounded and bounded away from zero.

4.3. *Remark.* Let m be an invariant measure on X and define $\mu(f; m) = \int \mu(f; x)\, dm(x)$, $f \in C(X)^{-1}$. Note that if m is ergodic, then this definition of $\mu(f; m)$ agrees with the definition given earlier. Then Schwartzman [Schw] calls μ an asymptotic cycle because it pairs orbits in X with cocycles. If X is a smooth manifold, and if the flow is generated by a smooth vector field Z on X, then μ is also given by the duality between $H^1(X, \mathbb{R})$ and $H_1(X, \mathbb{R})$, where these groups are, respectively, the first Čech cohomology and homology groups of X with real coefficients. More precisely, for an invertible $f \in C^1(X)$, one obtains a 1-form ω_f on X defined by the formula $\langle Y, \omega_f \rangle = Y(f)/f$ for any smooth vector field Y on X; i.e., $\omega_f = df/f$. This 1-form determines an element in $H^1(X, \mathbb{R})$ by de Rham's theorem. On the other hand, Z and m together determine a Ruelle–Sullivan class $[C] \in H_1(X, \mathbb{R})$ via the formula $\langle [C], [\omega] \rangle = (-1/2\pi i) \int_X \omega(Z)\, dm$, where ω is any closed 1-form on X and $[\omega]$ denotes the element of $H^1(X, \mathbb{R})$ it determines. We then have $\mu(f; m) = (-1/2\pi i) \int_X (f'(x)/f(x)) \, dm(x) = (-1/2\pi i) \int_X \omega_f(Z) \, dm(x) = \langle [C], [\omega_f] \rangle$ because $f' = Z(f)$.

4.4. *Remark.* Later, in Part II, we will require that $\tilde{\mu}$ be injective (cf. also, Theorem 1.5(iii)). Jerry Kaminker and Steve Hurder have shown us the following example where $\tilde{\mu}$ fails to be injective. Let X be the tangent sphere bundle of a compact Riemann surface Σ with genus g larger than 1. Then, from the Gysin sequence for an orientable sphere bundle, one concludes that the projection $P: X \to \Sigma$ induces an isomorphism $p^*: H^1(\Sigma, \mathbb{Z}) \to H^1(X, \mathbb{Z})$. Hence $H^1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ which is not zero. On the other hand, the $\tilde{\mu}$ associated with the horocyclic flow on X is the zero map. The reason for this is that the geodesic flow is transverse to the horocyclic flow and exponentially expanding. To be more precise, let T_s be the horocyclic flow and let S_t be the geodesic flow. Then $S_t T_s S_t = T_{\exp(\lambda t)s}$ for all s and t and some λ. It follows on the one hand that when viewed as acting on homology, $S_t T_s S_t$ expands $\tilde{\mu}$ by a factor of $\exp(\lambda t)$ while on the other, since $S_t T_s S_t$ is homotopic to T_s, $S_t T_s S_t$ must leave $\tilde{\mu}$ fixed. Hence $\tilde{\mu}$ must be zero. (See Arnold and Avez [AA] for facts about horocyclic and geodesic flows.) It seems to be a difficult problem to provide general conditions under which $\tilde{\mu}$ is injective.

4.5. Definition. If x is a quasi-regular point in X, we call the image of $H^1(X, \mathbb{Z})$ under $\tilde{\mu}(\cdot; x)$ the *module of the flow* $(X, \mathbb{R})$ determined by x. If the flow is strictly ergodic, all the modules are the same and we refer to their common value as the module of the flow. If $f \in C(X)$ and if x is a quasi-regular point in X, then the *module of f* determined by x is simply the module of the quotient flow determined by f and the image of x under the quotient map.

4.6. *Remark.* If the flow is almost periodic, and if $f \in C(X)$, then the module of f determined by $x \in X$ is simply the classical module of the almost periodic function obtained by restricting f to the orbit through x. This is easy to see on the basis of Chapter I of [JT], where, in effect, Jessen and Tornehave show that if f is an almost periodic function on $\mathbb{R}$ with spatial extension $\bar{f}$ living in $C(X)$, where X is a quotient of the Bohr group, then the module of f, which is the dual of X, coincides with the image of μ.

5.1. We write $A(X, \mathbb{R})$ for the space of functions $f \in C(X)$ such that for each x, the function of t, $f(x+t)$, lies in the usual Hardy space $H^\infty(\mathbb{R})$ consisting of boundary values of functions that are bounded and analytic in the upper half-plane. The following two facts arte shown in [M1]: (i) If $(X, \mathbb{R})$ is strictly ergodic, then $A(X, \mathbb{R})$ is a Dirichlet algebra on X; i.e., the space of functions $\{f+\bar{g} \mid f, g \in A(X, \mathbb{R})\}$ is uniformly dense in $C(X)$. (ii) If m is an invariant, ergodic, probability measure on X, then $A(X, \mathbb{R})$ is a weak-$*$ Dirichlet algebra on X. This means that the constant function 1 lies in $A(X, \mathbb{R})$, $\int fg\, dm = (\int f\, dm)(\int g\, dm)$, and $\{f+\bar{g} \mid f, g \in A(X, \mathbb{R})\}$ is weak-$*$ dense in $L^\infty(X)$.

5.2. Let $P_z(t) = (1/\pi) \operatorname{Im} z/(|z-t|^2)$ be the Poisson kernel for the upper half-plane $\{\operatorname{Im} z > 0\}$, and let $f \in A(X, \mathbb{R})$. We define a function F on $X \times \{\operatorname{Im} z > 0\}$ by the formula

$$F(x; z) = f * P_z(x),$$

where $f * P_z(x) = \int_{-\infty}^{\infty} f(x+t)\, P_z(t)\, dt$. It is immediate that F is continuous on $X \times \{\operatorname{Im} z > 0\}$ and satisfies the equation $F(x+t; z) = F(x; z+t)$ for all $t \in \mathbb{R}$. We may extend F to a continuous function on $X \times \{\operatorname{Im} z \geqslant 0\}$ by setting $F(x; z) = f(x+z)$ when $\operatorname{Im} z = 0$. Of course, for each x, $F(x; \cdot)$ is holomorphic in $\{\operatorname{Im} z > 0\}$. When x is fixed, we write $(\partial/\partial z)\, F(x, z_0)$ for the z-derivative of F evaluated at z_0. When z_0 is fixed in $\{\operatorname{Im} z > 0\}$, it is clear that $F(\cdot\,; z_0) \in C^1(X)$ and we write $F'(x; z_0)$ for the derivative (at x) of $F(\cdot\,; z_0)$ in the direction of the flow. Since $F(x+t; z) = F(x; z+t)$, it is evident that $F'(x; z) = (\partial/\partial z)\, F(x; z)$ on $X \times \{\operatorname{Im} z > 0\}$.

5.3. If we let $H(\operatorname{Im} z > 0)$ denote the space of all functions that are holomorphic in $\{\operatorname{Im} z > 0\}$, and if we give $H(\operatorname{Im} z > 0)$ the topology of uniform convergence on compact sets, then the map $x \to F(x; \cdot)$ is continuous from X to $H(\operatorname{Im} z > 0)$. This is immediate, based on estimates on the Poisson kernel. Consequently, $\{F(x; \cdot)\}_{x \in X}$ is a compact subset of $H(\operatorname{Im} z > 0)$.

5.4. If m is an invariant ergodic probability measure on X, then for m-almost all x, $\lim_{y \to \infty} F(x, iy) = \int_X f\, dm$ [M5, p. 360]. If $(X, \mathbb{R})$ is strictly ergodic, so that $A(X, \mathbb{R})$ is a Dirichlet algebra on X, then the maximal ideal space of $A(X, \mathbb{R})$ is homeomorphic to the cylinder $X \times [0, \infty]$ with the slice $X \times \{\infty\}$ identified to a point. The Gelfand transform of $f \in A(X, \mathbb{R})$, $\hat{f}$, is given by the formulae: $\hat{f}(x, y) = F(x; iy)$, $0 \leqslant y < \infty$, $\hat{f}(x, \infty) = \int f\, dm$ [M2].

5.5. Lemma. *If $(X, \mathbb{R})$ is minimal and if $f \in A(X, \mathbb{R})$ is not identically zero, then the zero function does not belong to $\{F(x; \cdot)\}_{x \in X}$.*

Proof. If for some x, $F(x; z)$ were identically zero in z, then since $F(x+t; z) = F(x; z+t)$ and since $f(x) = \lim_{y \to 0+} F(x; iy)$, we see that f vanishes on the orbit through x. Since f is continuous and $(X, \mathbb{R})$ is minimal, we conclude that $f \equiv 0$.

6. In Section 4 we discussed the mean motion of a nonvanishing continuous function on X. We now allow our functions to vanish at some points, but we restrict our attention to analytic functions, i.e., to functions in $A(X, \mathbb{R})$. For this, in turn, we must recall carefully how the argument of an analytic function behaves at zeros of the function.

The following notation and terminology is taken from [JT]. Let $G \subseteq \mathbb{C}$ be a domain and let $L \subseteq G$ be a line or line segment oriented so that we may speak of the right and left side of L. Also, let f be holomorphic on G. The *left argument* of f along L, $\arg^- f$, is defined by choosing an arbitrary branch of $\arg f(z_0)$ at a point $z_0 \in L$, where $f(z_0) \neq 0$ and then extending by continuity throughout L except at the zeros of f in such a way that when z passes a zero of order p in the positive direction, $\arg^- f$ jumps by $-p\pi$. The *right argument* of f, $\arg^+ f$, is defined in exactly the same fashion as $\arg^- f$ except that when passing a zero of order p in the positive direction, $\arg^+ f$ jumps by $+p\pi$. At a zero z_0 of order p, we set $\arg^\pm f(z_0) = \frac{1}{2}(\arg^\pm f(z_0+0) + \arg^\pm f(z_0-0))$. Evidently, $\arg^\pm f$ are defined only up to additive integral multiples of 2π, and if f has no zeros on L, then $\arg^\pm f$ agree and coincide with a continuous branch of the argument of f defined on L in the usual way. We set $\operatorname{Arg} f = \frac{1}{2}(\arg^+ f + \arg^- f)$. This function is continuous on L, is defined only modulo π, and satisfies the equation

$$f(z) = \rho(z)\, e^{2\pi i \operatorname{Arg} f(z)}$$

on L, where $\rho(z) = \pm |f(z)|$ and the sign is fixed on each subinterval of L where f does not vanish.

If points z_1 and z_2 on L are chosen so that the direction from z_1 to z_2 coincides with the positive direction of L, then the differences,

$$\arg^\pm f(z_2) - \arg^\pm f(z_1),$$

are called the *variations* of the argument along the right and left sides of L from z_1 to z_2. These quantities are independent of the branches of $\arg^\pm f$ used to define them and, evidently,

$$\arg^- f(z_2) - \arg^- f(z_1) \leqslant \arg^+ f(z_2) - \arg^+ f(z_1).$$

7. The next theorem is modeled on Theorem 3 of [JT]. We require the following notation. If s_1 and s_2 are arbitrary real numbers satisfying the inequality $s_1 < s_2$, and if y_1 and y_2 are nonnegative numbers satisfying the inequality $y_1 < y_2$, then we write $R(s_1, s_2, y_1, y_2)$ for the rectangle $\{z \mid s_1 < \operatorname{Re}(z) < s_2,\ y_1 < \operatorname{Im}(z) < y_2\}$. If $f \in A(X, \mathbb{R})$, we write $N(x; s_1, s_2, y_1, y_2)$ for the number of zeros of the associated function $F(x; \cdot)$ in $R(s_1, s_2, y_1, y_2)$ counted with multiplicity.

7.1. THEOREM [JT, Theorem 3]. *Assume that* $(X, \mathbb{R})$ *is minimal and that* f *is nonzero element of* $A(X, \mathbb{R})$. *Let* F *be the function associated with* f *in Section* 5.2. *Choose four numbers* y_1, y_2, y_3, *and* y_4 *satisfying* $0 \leqslant y_1 < y_2 < y_3 < y_4 \leqslant \infty$, *and let* d *satisfy* $0 \leqslant d \leqslant \min\{y_2 - y_1, y_4 - y_3\}$.

(i) *There is a number* N *such that for all* $t \in \mathbb{R}$ *and all* $x \in X$,

$$\begin{aligned} &N(x; t-(d+1/2), t+(d+1/2), y_2, y_3) \\ &\quad = N(x+t; -(d+1/2), (d+1/2), y_2, y_3) \leqslant N. \end{aligned}$$

(ii) *For each* $r > 0$, *there is a constant* m *depending only on* r *and* f, *not on* x, *such that* $|F(x; z)| \geqslant m$ *for all* $x \in X$, *and all* z *in the strip* $y_2 \leqslant \operatorname{Im} z \leqslant y_3$ that satisfy the inequality $|z - z_0(x)| \geqslant r$ for any zero $z_0(x)$ *of* $F(x; \cdot)$.

(iii) *There is a constant* k, *depending only on* f, *such that if* $z_1(x)$, $z_2(x), ..., z_{N(x)}(x)$, $N(x) \leqslant N$, *are the zeros of* $F(x; \cdot)$ *in* $R(t-(d+1/2)$, $t+(d+1/2), y_2, y_3)$, *counted with multiplicity, and if* $F^*(x; z)$ *is defined to be*

$$\frac{F(x; z)}{\prod_{n=1}^{N(x)} (z - z_n(x))},$$

then $|F^*(x; z)| \geqslant k$ on $X \times R(t-1/2, t+1/2, y_2, y_3)$.

(iv) *For each* $l > 0$ *there is a positive constant* $v = v(l)$ *such that the variation of the argument of* $F(x; \cdot)$ *along the left or right of any line segment of length less than* l *situated in the upper half-plane is less than or equal to* v.

Proof. By Lemma 5.5, $\{F(x; \cdot)\}_{x \in X}$ is a compact family in $H(\operatorname{Im} z > 0)$ that does not contain the zero function. Thus the proof for Theorem 3 of [JT] applies almost word for word to our setup for, as is shown in [JT], their result is simply a grand assertion about compact families in $H(\operatorname{Im} z > 0)$. Nothing more needs to be added.

7.2. *Remark.* Because Theorem 7.1 really is a statement about compact families in $H(\operatorname{Im} z > 0)$, we conclude that if $\{f_\lambda)_{\lambda \in \Lambda}$ is a compact set in $A(X, \mathbb{R})$, not containing the zero function, and if for each λ, F_λ is associated to f_λ as in Section 5.2, then we may choose the constants N, m, k, and v in Theorem 7.1 to be independent of $\lambda \in \Lambda$. In particular, if $\{f_n\}_{n=1}^{\infty}$ is a sequence in $A(X, \mathbb{R})$ converging uniformly to f in $A(X, \mathbb{R})$ and if neither f nor any term in the sequence is identically zero, then we may choose the constants in Theorem 7.1 to work for every f_n and f.

8.1. Fix $f \in A(X, \mathbb{R})$, and let F be associated with f as in Section 5.2. We follow Jessen and Tornehave in defining various mean motions of $F(x;\cdot)$ along horizontal lines. For $x \in X$ and $y > 0$ we set

$$\left\{\begin{matrix} \underline{\mu}^-(f; x, y) \\ \bar{\mu}^-(f; x, y) \end{matrix}\right\} = \frac{1}{2\pi} \lim_{T-S\to\infty} \begin{matrix} \inf \\ \sup \end{matrix} \left\{ \frac{\arg^- F(x; T+iy) - \arg^- F(x; S+iy)}{T-S} \right\}$$

and we set

$$\left\{\begin{matrix} \underline{\mu}^+(f; x, y) \\ \bar{\mu}^+(f; x, y \end{matrix}\right\} = \frac{1}{2\pi} \lim_{T-S\to\infty} \begin{matrix} \inf \\ \sup \end{matrix} \left\{ \frac{\arg^+ F(x; T+iy) - \arg^+ F(x; S+iy)}{T-S} \right\}.$$

These quantities are called, respectively, the *lower and upper, left and right mean motions of* $F(x;\cdot)$ *along the line* $z = iy$. They are clearly invariant functions, i.e., $\underline{\mu}^-(f; x+t, y) = \underline{\mu}^-(f; x, y)$ and likewise for the others, and the following inequalities are valid for all $x \in X$, and $y > 0$:

$$\underline{\mu}^-(f; x, y) \leqslant \left\{\begin{matrix} \underline{\mu}^+(f; x, y) \\ \bar{\mu}^-(f; x, y) \end{matrix}\right\} \leqslant \bar{\mu}^+(f; x, y).$$

If $\underline{\mu}^-(f; x, y) = \bar{\mu}^-(f; x, y)$, we say the *left mean motion* of $F(x;\cdot)$ exists along the line $z = iy$, and we denote it by $\mu^-(f; x, y)$. Similarly, if $\underline{\mu}^+(f; x, y) = \bar{\mu}^+(f; x, y)$, we say the *right mean motion* of $F(x;\cdot)$ exists along $z = iy$, and we denote it by $\mu^+(f; x, y)$. Evidently, if $\mu^\pm(f; x, y)$ exist and if we set $\mu^+(f; x, y) = \frac{1}{2}(\mu^+(f; x, y) + \mu^-(f; x, y))$, then

$$\mu(f; x, y) = \lim_{T-S\to\infty} \left\{ \frac{\operatorname{Arg}(F(x; T+iy)) - \operatorname{Arg}(F(x; S+iy))}{2\pi(T-S)} \right\}.$$

We call $\mu(f; x, y)$, when it exists, the *mean motion* of $F(x;\cdot)$ along the line $z = iy$.

8.2. Observe that if $F(x; iy) \neq 0$ on X, and if x is a quasi-regular point, then by Section 4 (see Lemma 4.1 and Remark 4.2 in particular), we have $2\pi i\mu(f; x, y) = \int_X (F'(\omega; iy)/F(\omega; iy))\, dm_x(\omega) = \int_X ((\partial/\partial z)\, F(\omega; iy)/F(\omega; iy))\, dm_x(\omega)$, where m_x is the probability measure associated with x as in Section 3.

8.3. Fix $0 < y_1 < y_2 < \infty$, and let $x \in X$. We define

$$\left.\begin{matrix} \underline{H}(x; y_1, y_2) \\ \bar{H}(x; y_1, y_2) \end{matrix}\right\} = \lim_{T-S\to\infty} \begin{matrix} \inf \\ \sup \end{matrix} \frac{N(x; S, T, y_1, y_2)}{T-S}.$$

These quantities are called, respectively, the *lower* and *upper relative frequency of zeros* of $F(x;\cdot)$ in the strip $y_1 < \operatorname{Im} z < y_2$. Note that by

Theorem 7.1(i), if $(X, \mathbb{R})$ is minimal or if $(X, \mathbb{R})$ is not minimal but f vanishes on no orbit, these quantities are finite for each x and each choice of y_1 and y_2. If $\underline{H}(x; y_1, y_2) = \bar{H}(x; y_1, y_2)$, then we call the common value *the relative frequency of zeros of* $F(x; \cdot)$ in the strip $y_1 < \operatorname{Im} z < y_2$ and we denote it by $H(x; y_1, y_2)$. Note that $\underline{H}$ and $\bar{H}$ are invariant functions and so is H, when it is defined.

9.1. LEMMA. *Fix x and suppose first that $F(x; \cdot)$ has no zeros on the boundary ∂R of the rectangle $R(s_1, s_2, u, v)$. Then*

$$\begin{aligned}
2\pi N(x; s_1, s_2, u, v) &= \frac{1}{i}\left\{\int_{s_1}^{s_2} \frac{F'(x+s; iu)}{F(x+s; iu)}\, ds - \int_{s_1}^{s_2} \frac{F'(x+s; iv)}{F(x+s; iv)}\, ds\right. \\
&\quad \left. + \int_u^v \frac{\partial/\partial z\, F(x+s_2; it)}{F(x+s_2; it)}\, dt - \int_u^v \frac{\partial/\partial z\, F(x+s_1; it)}{F(x+s_1; it)}\, dt\right\} \\
&= \int_{s_1}^{s_2} \frac{\partial \arg(F(x+s; iu))}{\partial s}\, ds - \int_{s_1}^{s_2} \frac{\partial \arg(F(x+s; iv))}{\partial s}\, ds \\
&\quad + \operatorname{Im} \int_u^v \left[\frac{\partial/\partial z\, F(x+s_2; it)}{F(x+s_2; it)} - \frac{\partial/\partial z\, F(x+s_1; it)}{F(x+s_1; it)}\right] dt \qquad (9.1) \\
&= -\int_{s_1}^{s_2} \frac{\partial \log|F(x+s; iu)|}{\partial u}\, ds - \int_{s_1}^{s_2} \frac{\partial \log|F(x+s; iv)|}{\partial v}\, ds \\
&\quad + \operatorname{Im} C(x; s_1, s_2, u, v), \qquad (9.2)
\end{aligned}$$

where

$$C(x; s_1, s_2, u, v) = \int_u^v \left[\frac{\partial/\partial z\, F(x+s_2; it)}{F(x+s_2; it)} - \frac{\partial/\partial z\, F(x+s_1; it)}{F(x+s_1; it)}\right] dt.$$

If $F(x; \cdot)$ is allowed to have zeros on the horizontal segments of ∂R only, then

$$\begin{aligned}
2\pi N(x; s_1, s_2, u, v) &= (\arg^- F(x; s_2 + iu) - \arg^- F(x; s_1 + iu)) \\
&\quad - (\arg^+ F(x; s_2 + iv) - \arg^+ F(x; s_1 + iv)) \\
&\quad + \operatorname{Im} C(x; s_1, s_2, u, v). \qquad (9.3)
\end{aligned}$$

Proof. Equations (9.1) and (9.2) are immediate consequences of the argument principle and the Cauchy–Riemann equations. Equation (9.3) results from the argument principle and the definition of $\arg^{\pm}$.

9.2. *Remark.* Observe that when $F(x; \cdot)$ does not vanish on the vertical sides on ∂R, Im $C(x; s_1, s_2, u, v)$ is the variation of the argument of $F(x; \cdot)$ along these sides. By Theorem 7.1(iv), this variation is bounded by a quantity depending only on $v-u$; i.e., it does not depend on x, s_1, or s_2, so long as $F(x; \cdot)$ does not vanish on the segments $[s_1+iu, s_1+iv]$ and $[s_2+iu, s_2+iv]$. This fact will be used several times in the sequel, but we will also need a variation, Lemma 9.4, that may be used when this observation does not apply.

9.3. *Remark.* Recall that a subset E of $\mathbb{R}$ is called *relatively dense* in case there is an $l>0$ such that every interval of length l in $\mathbb{R}$ meets E. Equivalently, E is relatively dense if and only if there is an $l>0$ such that $E+[0, l]=\mathbb{R}$. We note that the liminf and limsup in Section 8.3 are unchanged if S and T are restricted to lie in a relatively dense subset of $\mathbb{R}$. This is clear from Theorem 7.1(i).

9.4. LEMMA. *Assume that $(X, \mathbb{R})$ is minimal and let $0<y_1<y_2<\infty$ and $x_0 \in X$ all be fixed. Then there is a $\delta>0$ and a neighborhood U of x_0 such that for each $x \in U$, the set $\{s \in \mathbb{R} \mid |F(x; s+iy)| \geqslant \delta, y_1 \leqslant y \leqslant y_2\}$ is relatively dense in $\mathbb{R}$.*

Proof. Since $F(x_0; \cdot)$ is not identically zero and since F is continuous on $X \times (\operatorname{Im} z>0)$, it follows that there is an $s_0 \in \mathbb{R}$, a $\delta>0$, and a neighborhood U of x_0 such that

$$|F(x; s_0+iy)| \geqslant \delta$$

for all $x \in U$, and $y_1 \leqslant y \leqslant y_2$. Since the flow is assumed to be minimal, it is pointwise almost periodic [Ell]. This means that if $\tilde{E}_x = \{s \in \mathbb{R} \mid x+s \in U\}$, then $\tilde{E}_x$ is relatively dense for each $x \in U$. But then $E_x \equiv \tilde{E}_x + s_0$ is relatively dense and if $s=r+s_0 \in E_x$, then $|F(x; s+iy)| = |F(x+r; s_0+iy)| \geqslant \delta$. This completes the proof.

10. THEOREM (cf. [JT, Theorem 4]). *Suppose that $(X, \mathbb{R})$ is minimal, that $f \in A(X, \mathbb{R})$ is not the zero function, and that $0 \leqslant y_1 < y_2 < \infty$ are fixed. Then for all $x \in X$, we have*

$$\begin{aligned}
&(\underline{\mu}^-(f; x, y_2) - \bar{\mu}^+(f; x, y_1)) \\
&\quad \leqslant \underline{H}(x; y_1, y_2) \\
&\quad \leqslant \begin{Bmatrix} (\underline{\mu}^-(f; x, y_2) - \bar{\mu}^+(f; x, y_1)) \\ (\underline{\mu}^-(f; x, y_2) - \underline{\mu}^+(f; x, y_1)) \end{Bmatrix} \\
&\quad \leqslant \bar{H}(x; y_1, y_2) \leqslant (\bar{\mu}^-(f; x, y_2)) - \underline{\mu}^+(f; x, y_1)).
\end{aligned}$$

Proof. Fix $x \in X$. By Eq. (9.3) in Lemma 9.1, we may write $N(x; s_1, s_2, y_1, y_2) = (1/2\pi)(\arg^- F(x; s_2 + iy_1) - \arg^- F(x; s_1 + iy_1)) - (1/2\pi))(\arg^+ F(x; s_2 + iy_2) - \arg^+ F(x; s_1 + iy_2)) + (1/2\pi) \operatorname{Im} C(x; s_1, s_2, y_1, y_2)$, where s_1 and s_2 range over the set $E_x = \{s \in \mathbb{R} \mid F(x; z)$ does not vanish on $[s + iy_1, s + iy_2]\}$. By Lemma 9.4, E_x is relatively dense in $\mathbb{R}$. By Theorem 7.1(iv), or by using the formula for $C(x; s_1, s_2, y_1, y_2)$ in conjunction with Lemma 9.4, we see that

$$\sup\{|\operatorname{Im} C(x; s_1, s_2, y_1, y_2)| \mid s_1, s_2 \in E_x\}$$

is finite. So, if we divide through in the above equation by $s_2 - s_1$, take liminf's and limsup's with s_1 and s_2 ranging over E_x, and acknowledge Remark 9.3, we see that the desired inequalities are immediate consequences of the definitions.

11. At this point, we depart from the organization of Jessen and Tornehave. Our approach to the so-called *Jensen function* $\Phi(f; \cdot)$ of a function $f \in A(X, \mathbb{R})$ is based on the following theorem of Carl Carlson [Car]. For the sake of completeness and for the sake of discussion later, we present his proof.

11.1. THEOREM. *Suppose that* $(X, \mathbb{R})$ *is minimal and that* m *is an invariant, ergodic, probability measure on* X. *If* f *is a nonzero function in* $A(X, \mathbb{R})$, *then* $\log|f|$ *is integrable with respect to* m.

Proof. Assume, without loss of generality, that $|f| \leqslant 1$, so that $\log|f| \leqslant 0$. Also, fix $y > 0$, choose an open set $U \subseteq X$, and choose $\varepsilon > 0$ so that $|f * P_{iy}(x)| = |F(x; iy)| > \varepsilon$ for all $x \in U$. (Since no $F(x; \cdot)$ is identically zero, by Lemma 5.5, these choices are possible.) Next choose a nonnegative function $h \in C(X)$, $h \not\equiv 0$, that vanishes off U. Since $h \geqslant 0$ and $(X, \mathbb{R})$ is minimal, there is a constant M such that $h * P_{iy}(x) \geqslant M$ for all $x \in X$. Indeed, since $h > 0$ on an open set, and since $(X, \mathbb{R})$ is minimal, $h(x + t) > 0$ for all t in an open set in $\mathbb{R}$, depending on x, for each $x \in X$. Consequently, $h * P_{iy}(x) = \int_{-\infty}^{\infty} h(x + t)\, P_{iy}(t)\, dt > 0$ for each $x \in X$, and so the assertion follows from the compactness of X. One the other hand, by Jensen's inequality and the fact that for each $x \in X$ and $y > 0$, the map $f \to f * P_{iy}(x)$ is a representing measure for $A(X, \mathbb{R})$ (see Section 5.4), it follows that $\log|f * P_{iy}(x)| \leqslant (\log|f| * P_{iy})(x)$. Hence we have, by Fubini's theorem and the invariance of m, the following inequality that completes the proof:

$$M \int \log|f|\, dm \geqslant \int (h * P_{iy}) \log|f|\, dm$$

$$= \int h((\log|f|) * P_{iy})\, dm \geqslant \int h \log|f * P_{iy}|\, dm > -\infty.$$

11.2. *Remark.* Theorem 11.1 should be compared with a famous result of Helson and Lowdenslager [HL] (see [H] also). Let m be an invariant ergodic probability measure on X, and let $H^\infty(m)$ be the closure of $A(X, \mathbb{R})$ in the weak-$*$ topology of $L^\infty(m)$. Then there is a function $f \in H^\infty(m)$ such that $\log|f| \notin L^1(m)$. This was shown by Helson and Lowdenslager in the case when X is a quotient of the Bohr group [*HL*], however, the conclusion holds on any properly ergodic flow. The significant thing about the result is that for any $f \in H^\infty(m)$, f vanishes almost nowhere on X, and furthermore for almost all x, the function of t, $\log|f(x+t)|$, is integrable with respect to the measure $dt/(1+t^2)$ on $\mathbb{R}$. These things are consequences of the fact that $f \in H^\infty(m)$ if and only if $f \in L^\infty(m)$ and for almost all $x \in X$, the function of t, $f(x+t)$, lies in $H^\infty(\mathbb{R})$ [M3]. Thus we see that while a function in $H^\infty(m)$ cannot vanish too often, it can get very small; on the other hand, functions in $A(X, \mathbb{R})$ cannot get small very often. The importance of the problem with $H^\infty(m)$-functions for us is that it is the primary obstacle stopping us from developing a value distribution theory for functions in $H^\infty(m)$. That is, virtually all the other results in this part of the paper have measure theoretic analogues, or would have measure theoretic analogues, if it were the case that Theorem 11.1 is valid for functions $H^\infty(m)$.

It is interesting to note where the proof of Theorem 11.1 breaks down for functions in $H^\infty(m)$. Given $f \in H^\infty(m)$ and $y > 0$, one can find a set of positive measure U so that $|f * P_{iy}(x)|$ is bounded aways from zero a.e. on U. One can also choose $h \geqslant 0$ a.e., vanishing off U a.e.; namely, take $h = 1_{X \setminus U}$. One concludes from ergodicity that $h * P_{iy}(x) > 0$ for almost all $x \in X$ but one cannot conclude that $h * P_{iy}$ is essentially bounded below. This is all that is necessary to complete the proof. Thus, while the orbit of almost every point meets $X \setminus U$, there are many points whose orbits only just brush $X \setminus U$.

11.3. If $f \in A(X, \mathbb{R})$, then for each $y \geqslant 0$, $F(\cdot\,; iy)$ lies in $A(X, \mathbb{R})$ also. Hence if $(X, \mathbb{R})$ is minimal, and if m is an invariant, ergodic, probability measure, we may apply Theorem 11.1 to form the function

$$\Phi(f;\, y) = \frac{1}{2\pi} \int \log|F(x;\, iy)|\, dm(x).$$

Following Jessen and Tornehave, we call $\Phi(f;\,\cdot)$ the Jensen function of f (determined by the measure m). This function measures the density of zeros of each function $F(x;\,\cdot)$ in any prescribed strip and is closely connected to the mean motions of F calculated along horizontal lines. Shortly, we will assume that $(X, \mathbb{R})$ is strictly ergodic so that continual mention of the dependence on m will be unnecessary. The proof of the following theorem is modified from [L, p. 277].

11.4. THEOREM. *Suppose that $(X, \mathbb{R})$ is minimal and that m is an invariant ergodic probability measure on X. Then the Jensen function, $\Phi(f; y)$, of a nonzero function $f \in A(X, \mathbb{R})$ is a convex function of y.*

Proof. Fix $0 = y_1 < y_2 < y_3 < \infty$. It suffices to show that

$$\begin{aligned} &-[\Phi(f; y_2) - \Phi(f; y_1)](y_3 - y_2) + [\Phi(f; y_3) - \Phi(f; y_2)](y_2 - y_1) \\ &\quad = -[\Phi(f; y_1)(y_2 - y_3) + \Phi(f; y_2)(y_3 - y_1) + \Phi(f; y_3)(y_1 - y_2)] \end{aligned}$$

is nonnegative. To this end, apply the individual ergodic theorem and Theorem 11.1 to find an $x \in X$ such that

$$2\pi\Phi(f; y_k) = \lim_{s_2 - s_1 \to \infty} \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} \log|F(x+s; iy_k)|\, ds,$$

$k = 1, 2, 3$. For this x, apply Lemma 9.4 to find a relatively dense subset E of $\mathbb{R}$ such that

$$\sup\{|C(x; s_1, s_2, u, v)| \mid y_1 \leqslant u < v \leqslant y_3,\ s_1, s_2 \in E\}$$

is finite, where C is defined in Lemma 9.1. Then use Eq. (9.2) in Lemma 9.1 and integrate to obtain

$$\begin{aligned} &\frac{2\pi}{s_2 - s_1} \int_{y_2}^{y_3} \int_{y_1}^{y_2} N(x; s_1, s_2, u, v)\, du\, dv \\ &\quad = -\left\{\frac{1}{s_1 - s_2} \int_{s_1}^{s_2} [\log|F(x+s; iy_2)| - \log|F(x+s; iy_1)|]\, ds\right\} (y_3 - y_2) \\ &\quad\quad + \left\{\frac{1}{s_2 - s_1} \int_{s_1}^{s_2} [\log|F(x+s; iy_3)| - \log|F(x+s; iy_2)|]\, ds\right\} (y_2 - y_1) \\ &\quad\quad + \frac{1}{s_2 - s_1} \int_{y_2}^{y_3} \int_{y_1}^{y_2} C(x; s_1, s_2, u, v)\, du\, dv. \end{aligned}$$

By hypothesis, the limit as $s_2 - s_1 \to \infty$, $s_1, s_2 \in E$, of the right hand side of this equation is, except for a factor of 2π,

$$-[\Phi(f; y_2) - \Phi(f; y_1)](y_3 - y_2) + [\Phi(f; y_3) - \Phi(f; y_2)](y_2 - y_1).$$

Since the left hand side of the equation is nonnegative, the result follows.

11.5. *Remark.* In particular, we conclude that $\Phi(f; y)$, as a function of y, is continuous. We are unable to prove this directly. If we could, then the proof of Theorem 14.1 below would give another proof of Theorem 11.4 in the context of strictly ergodic flows.

12. Suppose that $f \in A(X, \mathbb{R})$ is not identically zero. We define the function $\Phi(f; x, \lambda, s_1, s_2, y)$ by the formula

$$\Phi(f; x, \lambda, s_1, s_2, y) = \frac{1}{2\pi(s_2 - s_1)} \int_{s_1}^{s_2} \log |F(x+s; iy)|_\lambda \, ds,$$

where $x \in X$, $s_1 < s_2$, $y > 0$ $\lambda \geqslant 0$, and where $|F(x; z)|_\lambda = \max\{|F(x; z)|, \lambda\}$. For $\lambda > 0$, it is clear that $\Phi(f; x, \lambda, s_1, s_2, y)$ is continuous in all variables and uniformly so in x and y, for y restricted to compact intervals. Also, this function is continuous when $\lambda = 0$. The reason is that $F(x+s; iy) = F(x; s+iy)$ is holomorphic in $z = s+iy$ for each x. So, by Theorem 7.1, when restricted to the rectangle $R(s_1, s_2, y_1, y_2)$ the function $F(x; \cdot)$ has a finite number of zeros that is bounded above independent of x. Moreover, if $z_0 = s_0 + iy_0$ is a zero of $F(x_0; \cdot)$ of multiplicity p in this rectangle, then a straightforward application of Rouché's theorem shows that we may find a neighborhood $U \times V$ of (x_0, z_0) and p continuous functions $z_i \colon U \to V$ satisfying $z_i(x_0) = z_0$, $i = 1, 2, ..., p$, such that

$$F(x; z) = \prod_{i=1}^{p} (z - z_i(x)) \, G(x; z)$$

on $U \times V$, where G is continuous, zero-free on $U \times V$, and for each $x \in U$, $G(x; \cdot)$ is holomorphic on V. To check the continuity of $\Phi(f; x, 0, s_1, s_2, y)$ it suffices to show that if $[s_0 - \varepsilon + iy_0, s_0 + \varepsilon + iy_0]$ is a small horizontal line segment lying entirely in V, then $\Phi(f; x, 0, s_0 - \varepsilon, s_0 + \varepsilon, y)$ is continuous on $U \times I$ where I is an open interval about y_0. We have $\log |F(x; z)| = \sum_{i=1}^{p} \log |z - z_i(x)| + \log |G(x; z)|$ in $U \times V$. Since $G(x; z)$ is zero-free in $U \times V$, its contribution to the desired integral does not affect the continuity. So it suffices to check the continuity of

$$\int_{s_0 - \varepsilon}^{s_0 + \varepsilon} \log |s + iy - z_j(x)| \, ds,$$

$j = 1, 2, ..., p$, as a function of $(x, y) \in U \times I$. Since this is an easy, albeit messy calculation based on integration by parts, we omit the details.

The individual ergodic theorem implies that for each y there is a null set of x *depending* on y such that for x in the complement,

$$\lim_{s_2 - s_1 \to \infty} \Phi(f; x, 0, s_1, s_2, y) = \Phi(f; y).$$

In order to prove our version of Jessen and Tornehave's Theorem 5 in [JT] that relates $\Phi(f; y)$ to the distribution of the zeros of F and to mean motions, we would like to find one x such that for *all* y in a given interval, $\lim_{s_2-s_1\to\infty}\Phi(f; x, 0, s_1, s_2, y)=\Phi(f; y)$. Unfortunately, at this stage, we are unable to do this without extra hypotheses. This is where we begin to assume that our flow is strictly ergodic. From now on, if we say that $(X, \mathbb{R})$ is strictly ergodic, then m will denote the unique invariant probability measure on X, and our Jensen functions will be computed with respect to this measure.

13. THEOREM (cf. [JT, Theorem 5]). *Suppose that* $(X, \mathbb{R})$ *is strictly ergodic and that* $f\in A(X, \mathbb{R})$ *is not identically zero. Then in any interval* $[y_1, y_2]$, *with* $y_1>0$, *we have*

$$\lim_{s_1-s_2\to\infty}\Phi(f; x, 0, s_1, s_2, y)=\Phi(f; y)$$

uniformly on $X\times[y_1, y_2]$.

Proof. Let $\Phi(f; \lambda, y)=(1/2\pi)\int_X \log|F(x; iy)|_\lambda\, dm(x)$ and consider the inequality

$$\begin{aligned}
&|\Phi(f; x, 0, s_1, s_2, y)-\Phi(f; y)|\\
&\quad\leqslant|\Phi(f; x, 0, s_1, s_2, y)-\Phi(s; x, \lambda, s_1, s_2, y)|\\
&\qquad+|\Phi(f; x, \lambda, s_1, s_2, y)-\Phi(f; \lambda, y)|\\
&\qquad+|\Phi(f; \lambda, y)-\Phi(f; y)|\\
&\quad=I_1+I_2+I_3.
\end{aligned}$$

Observe that as $\lambda\downarrow 0$, $\log|F(x; z)|_\lambda\downarrow\log|F(x; z)|$. By the monotone convergence theorem, we conclude that $\Phi(f; \lambda, y)\downarrow\Phi(f; y)$ for each y. Since $\Phi(f; \cdot)$ is continuous, by Theorem 11.4, and since each $\Phi(f; \lambda, \cdot)$ is also continuous, it follows that I_3 goes to zero as $\lambda\downarrow 0$ uniformly in $[y_1, y_2]$. For fixed $\lambda>0$, $\log|F(x; z)|_\lambda$ is continuous on $X\times(\operatorname{Im} z>0)$. Consequently, for each $\lambda>0$, $\lim_{s_2-s_1\to\infty} I_2=0$ uniformly on $X\times[y_1, y_2]$ by Lemma 3.3. Thus, it suffices to show that given $\varepsilon>0$, we can find $\lambda_0>0$ such that for $0<\lambda\leqslant\lambda_0$, for $s_2-s_1>1$, and for all $(x, y)\in X\times[y_1, y_2]$, the inequality $I_1<\varepsilon$ holds.

To this end, observe that since $|F(x; z)|_\lambda\geqslant|F(x; z)|$, what we want to show is that we may find $\lambda_0>0$ such that for $0<\lambda\leqslant\lambda_0$, $s_2-s_1>1$, $x\in X$, and $y_1\leqslant y\leqslant y_2$, we have $0\leqslant 2\pi(s_2-s_1)\, I_1=\int_{s_1}^{s_2}\log|F(x+s; iy)|_\lambda-\log|F(x+s; iy)|\, ds\leqslant 2\pi\varepsilon(s_2-s_1)$. Fix y_0 and y_3 so that $0<y_0<y_1<y_2<y_3<\infty$, and let $0<d\leqslant\min\{y_1-y_0, y_3-y_2\}$. By Theorem 7.1(ii), for each $r>0$, there is a $\lambda=\lambda(r)$ such that $|F(x; z)|_\lambda=|F(x; z)|$ for all $x\in X$

and all z in the strip $y_1 \leqslant \operatorname{Im} z \leqslant y_2$ satisfying $|z - z_0(x)| \geqslant r$ for any zero $z_0(x)$ of $F(x; \cdot)$. Note that once one λ with this property is found, any smaller λ will work too. Consequently, we may assume $\lambda < 1$. If we choose $r < d$, then by Theorem 7.1(i), there is a number N independent of r, $x \in X$, and $y \in [y_1, y_2]$, such that in every integral

$$J(x; t, y) \equiv \int_{t-1/2}^{t+1/2} \log|F(x+s; iy)|_\lambda - \log|F(x+s; iy)|\, ds,$$

the integrand is positive in at most N subintervals of $(t-\frac{1}{2}, t+\frac{1}{2})$ having total length $\leqslant N \cdot 2r$. Since $\lambda < 1$, in each of these subintervals, the integrand, $\log|F(x; s+iy)|_\lambda - \log|F(x; s+iy)|$, is dominated by $-\log^-|F(x; s+iy)|$, where $\log^- u = \min\{\log u, 0\}$. Note that a smaller choice of λ increases neither the number of intervals in $(t-\frac{1}{2}, t+\frac{1}{2})$ nor the upper bound $-\log^-|F(x; s+iy)|$ of the integrand in the intervals. Now we apply Theorem 7.1(iii) to conclude that there is an absolute constant k such that if $z_1(x), ..., z_{N(x)}(x)$, $N(x) \leqslant N$, are the zeros of $F(x; \cdot)$ in $R(t-(d+\frac{1}{2}), t+(d+\frac{1}{2}), y_1\, y_2)$, counted with multiplicity, and if

$$F^*(x; z) \equiv \frac{F(x; z)}{\prod_{n=1}^{N(x)} (z - z_n(x))},$$

then $|F^*(x; z)| \leqslant k$. It follows that

$$\begin{aligned} -\log^-|F(x; s+iy)| &\leqslant -\log^- k - \sum_{n=1}^{N(x)} \log^-|(s+iy) - z_n(x)| \\ &\leqslant -\log^- k - \sum_{n=1}^{N(x)} \log^-|s - s_n(x)|, \end{aligned}$$

where $s_n(x) = \operatorname{Re}(z_n(x))$. Thus in each of the intervals where the integrand in $J(x; t, y)$ is positive, the integrand is dominated by

$$-\log^- k - \sum_{n=1}^{N(x)} \log^-|s - s_n(x)|.$$

Since there are at most N intervals of total length $N \cdot 2r$, we conclude that $0 \leqslant J(x; t, y) \leqslant -(\log^- k) \cdot N \cdot 2r - N \int_{-Nr}^{Nr} \log^-|u|\, du$ independent of $x \in X$, $t \in \mathbb{R}$, and $y \in [y_1, y_2]$. Note, too, that once this inequality is satisfied for a choice of $\lambda < 1$, it is satisfied for any smaller positive λ. Now, the right-hand side of the inequality tends to zero with r. So we may choose r so that it is $\leqslant \pi\varepsilon$. For this choice of r, and corresponding choice of λ, let A be the greatest integer less than $s_2 - s_1 > 1$. Then for appropriate choices of $t_1 < t_2 < \cdots < t_{A+1}$,

$$2\pi(s_2 - s_1)\, I_1 = \int_{s_1}^{s_2} \log|F(x+s; iy)|_\lambda - \log|F(x+s; iy)|\, ds$$
$$\leqslant \sum_{i=1}^{A+1} J(x; t_i, y) \leqslant (2A)(\tfrac{1}{2}\varepsilon) \leqslant 2\pi(s_2 - s_1)\varepsilon$$

for all $x \in X$, $y \in [y_1, y_2]$, and any smaller λ. This completes the proof.

13.1. COROLLARY. *Suppose that $(X, \mathbb{R})$ is strictly ergodic and that $\{f_n\}_{n=1}^{\infty}$ is a sequence in $A(X, \mathbb{R})$ converging uniformly to $f \in A(X, \mathbb{R})$. If neither f nor any f_n is the zero function, then for each $y > 0$, $\lim_{n\to\infty} \Phi(f_n; y) = \Phi(f; y)$ and the convergence is uniform for y in any compact set.*

Proof. From the inequality

$$\begin{aligned}
&|\Phi(f_n; y) - \Phi(f; y)| \\
&\quad \leqslant |\Phi(f_n, y) - \Phi(f_n; x, 0, s_1, s_2, y)| \\
&\qquad + |\Phi(f_n; x, 0, s_1, s_2, y) - \Phi(f; x, 0, s_1, s_2, y)| \\
&\qquad + |\Phi(f; x, 0, s_1, s_2, y) - \Phi(f; y)|
\end{aligned}$$

we see that it clearly suffices to prove that

$$\lim_{s_2 - s_1 \to \infty} \Phi(f_n; x, 0, s_1, s_2, y) = \Phi(f_n; y)$$

uniformly in n. If we use the initial estimate in the proof of Theorem 13, we have $|\Phi(f_n; x, 0, s_1, s_2, y) - \Phi(f_n; y)| \leqslant I_{n,1} + I_{n,2} + I_{n,3}$, where the subscript n is to indicate the dependence on n of all the quantities in Theorem 13. As $\lambda \downarrow 0$, $\log|F_n(x; z)|_\lambda \downarrow |F(x; z)|$ uniformly in n by the continuous dependence of F on f. Hence $I_{n,3}$ goes to zero uniformly in n and $y \in [y_1, y_2]$. An appeal to Lemma 3.3 shows that for each $\lambda > 0$, $\lim_{s_2 - s_1 \to \infty} I_{n,2} = 0$ uniformly on $X \times [y_1, y_2] \times \mathbb{N}$. Finally, it suffices to show that given $\varepsilon > 0$, we can find $\lambda_0 > 0$ such that for $0 < \lambda \leqslant \lambda_0$, for $s_2 - s_1 > 1$, and for all $(x, y, n) \in X \times [y_1, y_2] \times \mathbb{N}$, the inequality $I_{n,1} < \varepsilon$ holds. As was shown in the proof of Theorem 13, everything rests on being able to choose the numbers $\lambda = \lambda(r)$, N, and k, guaranteed by Theorem 7.1, independent of n. Since such a choice is possible, by Remark 7.2, the proof is complete.

14. We come now to the main theorem of the first part of this paper,

an assertion relating Φ, μ, and H. It is an analogue of Jessen and Tornehave's Theorem 7.

14.1. THEOREM (cf. [JT, Theorem 7]). *If* $(X, \mathbb{R})$ *is strictly ergodic, and* $f \in A(X, \mathbb{R})$ *is not identically zero, then, for* $0 < y < \infty$ *and for all* $x \in X$, $\Phi'(f; y-0) \leqslant \underline{\mu}^-(f; x, y) \leqslant \left\{\begin{smallmatrix}\underline{\mu}^+(f; x, y)\\ \bar{\mu}^-(f; x, y)\end{smallmatrix}\right\} \leqslant \bar{\mu}^+(f; x, y) \leqslant \Phi'(f; y+0)$. *Furthermore, if* $0 < y_1 < y_2 < \infty$, *then for each* $x \in X$, $(\Phi'(f; y_2 - 0) - \Phi'(f; y_1 + 0)) \leqslant \underline{H}(x; y_1, y_2) \leqslant \bar{H}(x; y_1, y_2) \leqslant (\Phi'(f; y_2 + 0) - \Phi'(f; y_1 - 0))$.

Proof. First note that by Theorem 11.4, $\Phi(f; y)$ is convex and so has right- and left-hand derivatives at each point. These are increasing and, except for at most countably many points y, $\Phi'(f; y-0) = \Phi'(f; y+0)$; i.e., $\Phi'(f; y)$ exists for all but countable many y. By Theorem 10 and the basic inequalities satisfied by the various mean motions (Section 8.1), it suffices to prove that

$$\begin{aligned} &\Phi'(f; y-0) \leqslant \underline{\mu}^-(f; x, y) \\ &\bar{\mu}^+(f; x, y) \leqslant \Phi'(f; y+0) \end{aligned} \tag{14.1}$$

for all $y > 0$ and all $x \in X$.

We write $\Phi(f; x, s_1, s_2, y)$ for $\Phi(f; x, 0, s_1, s_2, y)$. By Theorem 13, $\lim_{s_2 - s_1 \to \infty} \Phi(f; x, s_1, s_2, y) = \Phi(f; y)$ uniformly for $x \in X$ and y in any prescribed compact interval, say $[y_1, y_2]$. Also, $\Phi(f; \cdot, \cdot, \cdot, \cdot)$ is a continuous function of all its variables. If x is fixed and if $F(x; z)$ does not vanish on the interval $[s_1 + iy, s_2 + iy]$, then by the Cauchy–Riemann equations, we see that

$$\frac{\partial}{\partial y} \Phi(f; x, s_1, s_2, y) = \frac{\arg F(x; s_2 + iy) - \arg F(x; s_1 + iy)}{2\pi(s_2 - s_1)}.$$

Moreover, for x, s_1, and s_2 fixed, there are at most finitely many y in $[y_1, y_2]$ such that this partial derivative fails to exist. Nevertheless, by the mean value theorem, at these points y, the partial derivatives have right and left limits by the formula

$$\begin{aligned} &\frac{\partial}{\partial y} \Phi(f; x, s_1, s_2, y \pm 0) \\ &\qquad = \frac{\arg^{\pm}(F(x; s_2 + iy)) - \arg^{\pm}(F(x; s_1 + iy))}{2\pi(s_2 - s_1)}. \end{aligned} \tag{14.2}$$

Continue to hold x fixed. By Lemma 9.4, there is an $m > 0$ and a relatively dense subset E_x of $\mathbb{R}$ such that $|F(x; s + iy)| \geqslant m$ for all $s \in E_x$. If K is an upper bound for $|(\partial/\partial z) F(x; z)|$ in the strip $y_1 \leqslant \operatorname{Im} z \leqslant y_2$, and if $s_1, s_2 \in E_x$, then by Eq. (9.2) in Lemma 9.1, we have $|\operatorname{Im} C(x; s_1, s_2, y_1, y_2)| \leqslant (y_2 - y_1)(2K/m)$.

Note that Eq. (9.3) in Lemma 9.1 becomes

$$\begin{aligned} N&(x; s_1, s_2, y_1, y_2) \\ &= \left(\frac{\partial}{\partial y} \Phi(f; s_1, s_2, y_2 - 0) - \frac{\partial}{\partial y} \Phi(f; x, s_1, s_2, y_2 + 0)\right) \\ &\quad + \frac{1}{2\pi} \operatorname{Im} C(x; s_1, s_2, y_1, y_2). \end{aligned}$$

Following Jessen and Tornehave, we let $\Phi_1(f; x, s_1, s_2, y) = \Phi(f; x, s_1, s_2, y) + (1/(s_2 - s_1))(K/m)\, y^2$. Then our formula for $N(x; s_1, s_2, y_1, y_2)$ yields

$$\begin{aligned} &\frac{N(x; s_1, s_2, y_1, y_2)}{s_2 - s_1} \\ &\quad = \left(\frac{\partial}{\partial y} \Phi_1(f; x, s_1, s_2, y_2 - 0)\right. \\ &\qquad \left. - \frac{\partial}{\partial y} \Phi_1(f; x, s_1, s_1, y_1 + 0) + R_1(x; s_1, s_2, y_1, y_2)\right), \end{aligned}$$

where $-((y_2 - y_1)/(s_2 - s_1))(4K/m) \leqslant R_1(x; s_1, s_2, y_1, y_2) \leqslant 0$, when $s_1, s_2 \in E_x$. Since N is a nonnegative function, we conclude from this that for all $x \in X$, $s_1, s_2 \in E_x$, and $y > 0$,

$$\frac{\partial}{\partial y} \Phi_1(f; x, s_1, s_2, y_1 + 0) \leqslant \frac{\partial}{\partial y} \Phi_1(f; x, s_1, s_2, y_2 - 0),$$

i.e., as a function of y, Φ_1 is convex as long as $s_1, s_2 \in E_x$. Also, from the definition of Φ_1 and Theorem 13, we see that

$$\lim_{s_2 - s_1 \to \infty} \Phi_1(f; x, s_1, s_2, y) = \Phi(f; y),$$

and so

$$\begin{aligned} \Phi'(f; y - 0) &\leqslant \liminf_{s_2 - s_1 \to \infty, s_1 - s_2 \in E_x} \frac{\partial}{\partial y} \Phi_1(f; x, s_1, s_2, y - 0) \\ &\leqslant \limsup_{s_2 - s_1 \to \infty, s_1, s_2 \in E_x} \frac{\partial}{\partial y} \Phi_1(f; x, s_1, s_1, y + 0) \\ &\leqslant \Phi'(f; y + 0). \end{aligned} \tag{14.3}$$

(These inequalities follow from the fact that if a sequence of convex functions $\{f_n\}_{n=1}^{\infty}$ converges pointwise on an interval (a, b) to a function f, then f is convex, of course, and $f'(x-0) \leqslant \underline{\lim}\, f_n'(x-0) \leqslant \overline{\lim}\, f_n'(x+0) \leqslant f'(x+0)$ for all $x \in (a, b)$.) From the definition of Φ_1, we see that we may replace it by Φ in inequality (14.3). But then, from equation (14.2) we conclude from this inequality that the inequalities (14.1) hold. This completes the proof.

14.2. *Remark.* Theorem 14.1 does not give another proof that $\Phi(f; \cdot)$ is convex (under the hypothesis that $(X, \mathbb{R})$ is strictly ergodic) because the proof rests on Theorem 13 which in turn uses the fact that $\Phi(d; \cdot)$ is convex. As we noted in Remark 11.5, Theorem 14.1 would give a new proof that $\Phi(f; \cdot)$ is convex if there were a way to tell in advance that $\Phi(f; \cdot)$ is continuous in y.

We list several corollaries of Theorem 14.1 that are quite analogous to statements in Section 46 of [JT]. The proofs are immediate and so they will be omitted. The blanket hypotheses are those of Theorem 14.1: $(X, \mathbb{R})$ is strictly ergodic, and $f \in A(X, \mathbb{R})$ is not identically zero.

14.3. COROLLARY. *If $\Phi(f; \cdot)$ is differentiable at $y > 0$, then the mean motions $\mu^+(f; x, y)$ and $\mu^-(f; x, y)$ both exist and equal $\Phi'(f; y)$ for all $x \in X$.*

14.4. COROLLARY. *If $\Phi(f; \cdot)$ is differentiable at the points y_1 and y_2, $0 < y_1 < y_2$, then the relative frequency $H(x; y_1, y_2)$ of the zeros of $F(x; \cdot)$ exists for each x and is given by*

$$H(x; y_1, y_2) = (\Phi'(f; y_2) - \Phi'(f; y_1)). \tag{14.4}$$

In particular, $H(x; y_1, y_2)$ is constant in x.

Following Jessen and Tornehave, we will refer to Eq. (14.4) as *Jensen's formula.*

14.5. COROLLARY. *For each $y > 0$ and every $x \in X$ we have*

$$(\Phi'(f; y+0) - \Phi'(f; y-0)) = \lim_{\varepsilon \to 0} \underline{H}(x; y-\varepsilon, y+\varepsilon).$$

In particular, $\Phi(f; \cdot)$ is differentiable at y if and only if $\lim_{\varepsilon \to 0} \underline{H}(x; y-\varepsilon, y+\varepsilon) = 0$ for any x.

14.6. *Remark.* In [JT, Chap. V], it is shown (for almost periodic flows) that the inequalities that exist between the Jensen function of a function and its mean motions; i.e., the inequalities asserted by Theorem 14.1, are sharp. Explicitly, given six numbers satisfying

$$d^- \leqslant \underline{c}^- \leqslant \left\{ \begin{matrix} \underline{c}^+ \\ \bar{c}^- \end{matrix} \right\} \leqslant \bar{c}^+ \leqslant d^+,$$

a quotient X of the Bohr group (not isomorphic to $\mathbb{T}$), and an arbitrary $y > 0$, it is possible to find a function $f \in A(X, \mathbb{R})$ (in fact, for each $x \in X$, the function of t, $f(x+t)$, extends to be entire) so that (at a certain of x), $\Phi'(f; y-0) = d^-$, $\underline{\mu}^{\pm}(f; x, y) = \underline{c}^{\pm}(f; x, y) = \bar{c}^{\pm}$, and $\Phi'(f; y+0) = d^+$. The proof involves almost periodicity in an essential way and we wonder if the result is true for an arbitrary (nonperiodic) strictly ergodic flow.

15. When $(X, \mathbb{R})$ is strictly ergodic, the maximal ideal space of $A(X, \mathbb{R})$ (see Section 5.4) is sometimes referred to as a big disc. Accordingly, for $0 \leqslant y_1 < y_2 \leqslant \infty$, we refer to the subset $X \times (y_1, y_2)$ of the maximal ideal space of $A(X, \mathbb{R})$ as an annulus. The following theorem is an analogue of Jessen and Tornehave's Theorem 8.

15.1. THEOREM (cf. [JT, Theorem 8]). *Suppose that $(X, \mathbb{R})$ is strictly ergodic and that $f \in A(X, \mathbb{R})$ is not identically zero. Then F is zero-free on the annulus $X \times (y_1, y_2)$ if and only if $\Phi(f; \cdot)$ is linear on (y_1, y_2). In this case, for each y, $y_1 < y < y_2$, the mean motion of $F(\cdot; iy)$ along any orbit exists and is given by the formula*

$$\mu((F(\cdot; iy)) = \Phi'(f; y).$$

In particular, on any linearity interval, the derivative of the Jensen function of a function in $A(X, \mathbb{R})$ must lie in the module of the flow (see Section 4.5).

Proof. If $F(x; iy) \neq 0$ for all $(x, y) \in X \times (y_1, y_2)$ then $\bar{H}(x; y_1, y_2) = 0$ for all x. Consequently, by Theorem 14.1, $\Phi'(f; y_1 + 0) = \Phi'(f; y_2 - 0)$, showing that $\Phi'(f; \cdot)$ is linear on (y_1, y_2). If $F(x_0; iy_0) = 0$, then there is an r, $0 < r < \min\{|y_0 - y_1|, |y_0 - y_2|\}$ such that $F(x_0; z)$ does not vanish on the circle $|z - iy_0| = r$. If $m = \inf\{|F(x_0; z)| \mid |(z - iy_0)| = r\}$ and if U is a neighborhood of x_0 such that

$$|F(x_0; z) - F(x; z)| < m$$

for all $x \in U$ and all z, $|z - iy_0| = r$, then, by Rouché's theorem, each of the functions $F(x; \cdot)$, $x \in U$, vanishes somewhere in the interior of $|z - iy_0| = r$. In particular, for each s in the set $E_{x_0} \equiv \{s \mid x_0 + s \in U\}$, $F(x_0 + s; \cdot) = F(x_0, s + \cdot)$ vanishes at some point inside $|z - iy_0| = r$. But E_x is relatively dense since $(X, \mathbb{R})$, being minimal, is pointwise almost periodic. Consequently, we find that $F(x_0; \cdot)$ vanishes infinitely often in the strip $y_0 - r < \operatorname{Im} z < y_0 + r$ at points whose abscissae form a relatively dense subset of $\mathbb{R}$. It follows that $0 < \underline{H}(x_0; y_0 - r, y_0 + r)$. By Theorem 14.1, then, $\Phi'(f; y_0 - r) < \Phi'(f; y_0 + r)$. This shows that $\Phi(f; \cdot)$ is not linear on (y_1, y_2).

Suppose, now, $y_1 < y < y_2$. On the one hand, by what we just showed, $F(\cdot; iy)$ never vanishes on X. By Section 4, then, $F(\cdot; iy)$ has a mean motion along any orbit. On the other hand, since $\Phi'(f; \cdot)$ is constant in (y_1, y_2), we conclude from Theorem 14.1, that this mean motion must be $\Phi'(f; y)$. That completes the proof.

15.2. *Remark.* Jessen and Tornehave showed that if X is either the dual of a subgroup of the rationals or the dual of a subgroup of $\mathbb{R}$ generated by finitely or countably many rationally independent numbers, and if $\mathbb{R}$ acts on X in the usual way, then almost any convex function Φ on $(0, \infty)$ can serve as the Jensen function of some function in $A(X, \mathbb{R})$. The only restrictions on Φ are that on each linearity interval, Φ' must lie in the module of $(X, \mathbb{R})$ and, in case X is dual to a subgroup of $\mathbb{R}$ generated by rationally independent numbers, the linearity intervals of Φ that are contained in any compact subset of $(0, \infty)$ must be finite in number. It would be very interesting to see if a similar fact is true for arbitrary flows. Theorem 15.1 shows, of course, that on linearity intervals, $\Phi'(f; y)$ must belong to the module of $(X, \mathbb{R})$. Also, one can easily see that it may be necessary to restrict how the linearity intervals are distributed. However, to show that these conditions are sufficient appears to require a detailed analysis of the zeros of functions in $A(X, \mathbb{R})$ that is, at present, beyond our scope. Many parts of the almost periodic analysis apply to the general setting, but there also arise topological problems that must be dealt with. We hope to investigate these matters in the future.

16. Consider, finally, the function $\Phi(f - \zeta; y)$, where $f \in A(X, \mathbb{R})$ and $0 < y < \infty$, as usual, and where $\xi \in \mathbb{C}$. By Corollary 13.1, $\Phi(f - \cdot\,; \cdot)$ is continuous on $\mathbb{C} \times (0, \infty)$. Also, by Theorem 11.4, for each $\zeta \in \mathbb{C}$, $\Phi(f - \zeta; \cdot)$ is convex. Consequently, there is for each ζ, a set $F_\zeta \subseteq (0, \infty)$, which is at most countable, such that for each $y \notin F_\zeta$, $(\partial/\partial y)\, \Phi(f - \zeta; y)$ exists. We would like to reverse the roles of y and ζ and conclude that for each y, there is a null set E_y in the plane (i.e., null with respect to planar Lebesgue measure, $\mathfrak{L}_2$) such that $(\partial/\partial y)\, \Phi(f - \zeta; y)$ exists for each $\zeta \notin E_y$. This, apparently, is not easy to show directly. It is the kind of problem one frequently encounters in probability theory and we shall use probabilistic arguments in the proof. These are modifications of arguments used by Borchsenius and Jessen [BJ] for similar purposes.

16.1. THEOREM. *Assume that* $(X, \mathbb{R})$ *is strictly ergodic and that* $f \in A(X, \mathbb{R})$ *is not identically zero. Then, for each* $y_0 > 0$, *there is a null set* $E_{y_0} \subseteq \mathbb{C}$ *such that for* $\zeta \notin E_{y_0}$,

$$\frac{\partial}{\partial y} \Phi(f-\zeta;\, y_0-0) = \frac{\partial}{\partial y} \Phi(f-\zeta;\, y_0-0);$$

i.e., the derivative of $\Phi(f-\zeta;\, y_0)$ *with respect to* y *exists for all* $\zeta \notin E_{y_0}$.

Proof. Fix s_1, s_2, y_1, and y_2, satisfying the conditions $s_1 < s_2$, and $0 < y_1 < y_2 < \infty$, and let $N(f-\zeta; x; s_1, s_2, y_1, y_2)$ be the number of zeros of $F(x; \cdot) - \zeta$ in the rectangle $R(s_1, s_2, y_1, y_2)$. This is, of course, the same as the number of points, z, in $R(s_1, s_2, y_1, y_2)$ such that $F(x; z) = \zeta$. Also, for Borel sets $E \subseteq \mathbb{C}$, we set $A(x; s_1, s_2, y_1, y_2; E) = \{z \in R(s_1, s_2, y_1, y_2) \mid F(x; z) \in E\}$. We define a finite (positive) Borel measure $v(x; s_1, s_2, y_1, y_2; \cdot)$ on $\mathbb{C}$ by the formula

$$v(x; s_1, s_2, y_1, y_2; E) = \frac{1}{s_1 - s_2} \int_E N(f-\zeta; x; s_1, s_2, y_1, y_2)\, d\mathfrak{L}^2(\zeta)$$

where, recall, $\mathfrak{L}^2$ is planar Lebesgue measure. Then by the area theorem,

$$\begin{aligned} &v(x; s_1, s_2, y_1, y_2; E) \\ &\quad = \frac{1}{s_2 - s_1} \int_{A(x; s_1, s_2, y_1, y_2; E)} \left| \frac{\partial}{\partial z} F(x; z) \right|^2 d\mathfrak{L}^2(z) \\ &\quad = \frac{1}{s_2 - s_1} \int_{A(x; s_1, s_2, y_1, y_2; E)} |F'(x; z)|^2\, d\mathfrak{L}^2(z). \end{aligned}$$

From the definition of $v(x; s_1, s_2, y_1, y_2; \cdot)$ it is easy to compute its Fourier–Stieltjes transform (alias characteristic function) and to see that it is given by the formula

$$\hat{v}(x; s_1, s_2, y_1, y_2; \lambda) = \frac{1}{s_1 - s_2} \int_{s_1}^{s_2} \int_{y_1}^{y_2} e^{i\lambda \cdot F(x+t; iy)}\, |F'(x+t; iy)|^2\, dy\, dt,$$

where, for two complex numbers α and β, $\alpha \cdot \beta$ denotes the *scalar* or *dot product* of α and β viewed as vectors in $\mathbb{R}^2$. By Lemma 3.3,

$$\lim_{s_2 - s_1 \to \infty} \hat{v}(x, s_1, s_2, y_1, y_2; \lambda) = \int_X \int_{y_1}^{y_2} e^{i\lambda \cdot F(x; iy)} |F'(x; iy)|^2\, dy\, dm(x)$$

for all x, y_1, y_2, and λ, and the convergence is uniform for y_1, y_2, and λ

restricted to compact sets. The limit is the Fourier–Stieltjes transform of the measure $v(y_1, y_2; \cdot)$ on $\mathbb{C}$ defined by the formula

$$v(y_1, y_2; E) = \int_{y_1}^{y_2} \int_X 1_{\{F(x; iy) \in E\}} |F'(x; iy)|^2 \, dm(x) \, dy. \tag{16.1}$$

(This is a straightforward calculation using the definition of v and Fubini's theorem.) By a well-known theorem in Fourier analysis we conclude that for each continuity set E of $v(y_1, y_2; \cdot)$; i.e., for each Borel set $E \subseteq \mathbb{C}$ such that $v(y_1, y_2; E^0) = v(y_1, y_2; \bar{E})$, where E^0 denotes the interior of E and $\bar{E}$ denotes the closure of E, we have

$$\lim_{s_1 - s_2 \to \infty} v(x; s_1, s_2, y_1, y_2; E) = v(y_1, y_2; E)$$

for all $x \in X$. Of course the continuity sets of $v(y_1, y_2; \cdot)$ depend on y_1 and y_2, in general, but $\mathbb{C}$ is a continuity set for every $v(y_1, y_2; \cdot)$. One should view $v(y_1, y_2; E)$ as representing the average number of times F falls on E along any orbit when $\operatorname{Im} z$ is constrained to satisfy $y_1 \leqslant \operatorname{Im} z \leqslant y_2$.

Now let

$$\begin{matrix} \liminf \\ \limsup \end{matrix} \left\{ \frac{1}{s_2 - s_1} N(f - \zeta; x, s_1, s_2, y_1, y_2) = \right\{ \begin{matrix} \underline{H}(f - \xi; x; y_1, y_2) \\ \bar{H}(f - \zeta; x; y_1, y_2). \end{matrix}$$

Then, by Theorem 14.1, we have

$$\begin{aligned} \left(\frac{\partial}{\partial y} \Phi(f - \zeta; y_2 - 0) - \frac{\partial}{\partial y} \Phi(f - \zeta; y_1 + 0) \right) & \\ \leqslant \underline{H}(f - \zeta; x; y_1, y_2) & \\ \leqslant \bar{H}(f - \zeta; x; y_1, y_2) & \\ \leqslant \left(\frac{\partial}{\partial y} \Phi(f - \zeta; y_2 + 0) - \frac{\partial}{\partial y} \Phi(f - \zeta; y_1 - 0) \right) & \end{aligned}$$

for each $x \in X$. From what we just observed, and from Fatou's theorem, we conclude that if E is a continuity set for $v(y_1, y_2; \cdot)$, then for each x,

$$\begin{aligned} \int_E \underline{H}(f - \zeta; x; y_1, y_2) \, d\Omega^2(\zeta) &\leqslant v(y_1, y_2; E) \\ &\leqslant \int_E \bar{H}(f - \zeta; x; y_1, y_2) \, d\Omega^2(\zeta). \end{aligned}$$

Thus for all continuity sets E of $\nu(y_1, y_2; \cdot)$ we have

$$\int_E \left(\frac{\partial}{\partial y}\Phi(f-\zeta; y_2-0) - \frac{\partial}{\partial y}\Phi(f-\zeta; y_1+0)\right) d\Omega^2(\zeta)$$
$$\leqslant \nu(y_1, y_2; E)$$
$$\leqslant \int_E \left(\frac{\partial}{\partial y}\Phi(f-\zeta; y_2+0) - \frac{\partial}{\partial y}\Phi(f-\zeta; y_1-0)\right) d\Omega^2(\zeta). \quad (16.2)$$

If we let $E = \mathbb{C}$, fix y, and write $y_1 = y - \varepsilon$ and $y_2 = y + \varepsilon$ for some $\varepsilon > 0$, we see from Eq. (16.1) that $\lim_{\varepsilon \to 0} \nu(y-\varepsilon, y+\varepsilon; \mathbb{C}) = 0$. Thus from the left-hand side of Eq. (16.2) and Fatou's theorem, we conclude that $(\partial/\partial y)\,\Phi(f-\zeta; y-0) = (\partial/\partial y)\,\Phi(f-\zeta; y+0)$ for almost all ζ. This completes the proof.

16.2. *Remark.* Now, because we know that $(\partial/\partial y)\,\Phi(f-\zeta; y_1+0) = (\partial/\partial y)\,\Phi(f-\zeta; y_1-0)$ for almost all ζ, and likewise for $(\partial/\partial y)\,\Phi(f-\zeta; y_2 \pm 0)$, we conclude that inequality (16.2) is actually an equality for each continuity set E of $\nu(y_1, y_2; \cdot)$. However, once we have established equality in (16.2) for all continuity sets E, it follows from general principles that equality in (16.2) holds for all Borel sets E without restriction. Thus we see that the average number of times F falls in a given Borel set E along each orbit, with z constrained so that $y_1 < \operatorname{Im} z < y_2$, is given by either side of (16.2). This generalizes Eq. (8) of [BJ, p. 108].

If we combine Theorem 16.1 with Corollaries 14.3 and 14.4, we obtain the following two corollaries. In them, the hypotheses are those of Theorem 16.1.

16.3. COROLLARY. *For each y and almost all ζ, where the exceptional null set may depend upon y, the mean motions $\mu^{\pm}(f-\zeta; x, y)$ both exist and coincide with $(\partial/\partial y)\,\Phi(f-\zeta; y)$ for all $x \in X$. In particular, $u(f-\zeta; x, y)$, defined in Section* 18.1, *exists and equals $(\partial/\partial y)\,\Phi(f-\zeta; y)$ for all $x \in X$.*

16.4. COROLLARY. *Let $0 < y_1 < y_2$ be fixed. Then there is a planar null set E depending only on y_1 and y_2 such that for each $\zeta \notin E$ and for each $x \in X$, the relative frequency of zeros of $F(x; \cdot) - \zeta$ in the strip $y_1 < \operatorname{Im} z < y_2$, $H(f-\zeta; x; y_1, y_2)$, exists and is given by the formula $H(f-\zeta; x; y_1, y_2) = (\partial/\partial y)\,\Phi(f-\zeta; y_2) - (\partial/\partial y)\,\Phi(f-\zeta; y_1)$. In particular, for almost all ζ, $H(f-\zeta; x; y_1, y_2)$ is constant in x.*

PART II

17. We begin our analysis of Toeplitz operators on flows by collecting together some basic (known) facts about crossed products that we will need. Throughout, $(X, \mathbb{R})$ will be a flow. As in Part I, we do not make any blanket assumption about $(X, \mathbb{R})$ being strictly ergodic, although our most definitive results are based on that assumption (and the assumption that the mean motion map $\tilde{\mu}$ is injective).

17.1. Let $C_c(X \times \mathbb{R})$ be the compactly supported, continuous, complex-valued functions on $X \times \mathbb{R}$. Then $C_c(X \times \mathbb{R})$ is a locally convex topological vector space under the algebraic operations of pointwise addition and scalar multiplication and the inductive limit topology, i.e., the topology of uniform convergence on compact subsets of $X \times \mathbb{R}$. In addition, $C_c(X \times \mathbb{R})$ becomes a topological $*$-algebra under the product and involution given by the formulae

$$a * b(x, t) = \int a(x, s)\, b(x+s, t-s)\, ds,$$

and

$$a^*(x, t) = \overline{a(x+t, -t)},$$

for $a, b \in C_c(X \times \mathbb{R})$. A *representation* of $C_c(X \times \mathbb{R})$ is simply a $*$-homomorphism of $C_c(X - \mathbb{R})$ into the bounded operators on a Hilbert space $\mathscr{H}$ that is continuous with respect to the inductive limit topology on $C_c(X \times \mathbb{R})$ and the weak operator topology on $\mathscr{L}(\mathscr{H})$, the algebra of bounded operators on $\mathscr{H}$. For $a \in C_c(X \times \mathbb{R})$, set $\|a\| = \sup\{\|\pi(a)\| \mid \pi$ is a representation of $C_c(X \times \mathbb{R})\}$. This supremum is finite and defines a C^*-norm on $C_c(X \times \mathbb{R})$ (see [R]). The completion of $C_c(X \times \mathbb{R})$ in this norm, denoted $C(X) \rtimes \mathbb{R}$, is therefore a C^*-algebra called the *transformation group C^*-algebra or the crossed product C^*-algebra* determined by $(X, \mathbb{R})$. If $(X, \mathbb{R})$ is minimal and not conjugate to the usual action of $\mathbb{R}$ on the circle $\mathbb{T}$ through rotation, then $C(X) \rtimes \mathbb{R}$ is simple (see [EH]). Consequently, in this case, every representation of $C(X) \rtimes \mathbb{R}$ is faithful.

17.2. There are basically two useful ways to specify a representation of $C(X) \rtimes \mathbb{R}$. One can either define it first directly on $C_c(X \times \mathbb{R})$ and then extend it to all of $C(X) \rtimes \mathbb{R}$ by continuity, or one can give a covariant representation. The latter is a pair (σ, V), where σ is a C^*-representation of $C(X)$, and $V = \{V_t\}_{t \in \mathbb{R}}$ is a unitary representation of $\mathbb{R}$ such that

$\sigma(\varphi_t) = V_t \sigma(\varphi) V_t^*$, for all $t \in \mathbb{R}$, $\varphi \in C(X)$, where $\varphi_t(x) = \varphi(x+t)$ [EH]. In this paper, we will primarily be interested in types of representations of $C(X) \rtimes \mathbb{R}$ denoted, in the notation of [R], by $\operatorname{Ind} \delta_x$, $x \in X$, and $\operatorname{Ind} m$, where δ_x is the point mass at x and m is an invariant ergodic probability measure on X. The Hilbert space of $\operatorname{Ind} \delta_x$ is, for every x, $L^2(\mathbb{R})$, and $\operatorname{Ind} \delta_x$ is given by the formula

$$(\operatorname{Ind} \delta_x(a))\xi)(t) = \int a(x+s, s)\, \xi(t+s)\, ds,$$

$a \in C_c(X \times \mathbb{R})$, $\xi \in L^2(\mathbb{R})$. The Hilbert space of $\operatorname{Ind} m$ is $L^2(X \times \mathbb{R})$, where $X \times \mathbb{R}$ is given the product measure determined by m and Lebesgue measure on $\mathbb{R}$, and the defining formula for $\operatorname{Ind} m$ is

$$(\operatorname{Ind} m(a)\xi)(x, t) = \int a(x, s)\, \xi(x+s, t-s)\, ds,$$

$a \in C_c(X \times \mathbb{R})$, $\xi \in L^2(X \times \mathbb{R})$. On the one hand, the covariant representation (σ^x, U^x) associated with $\operatorname{Ind} \delta_x$ is given by the formulae

$$(\sigma^x(\varphi)\xi)(s) = \varphi(x+s)\, \xi(s),$$

and

$$(U_t^x \xi)(s) = \xi(s-t),$$

$\varphi \in C(X)$, $\xi \in L^2(\mathbb{R})$; while on the other hand, the covariant representation (σ^m, U^m) associated with $\operatorname{Ind} m$ is given by similar formulae,

$$\sigma^m(\varphi)\, \xi(x, s) = \varphi(x)\, \xi(x, s),$$

and

$$(U_t^m \xi)(x, s) = \xi(x+t, t-s).$$

17.3. The von Neumann algebra generated by $\operatorname{Ind} m(C_c(X \times \mathbb{R}))$ is denoted $L^\infty(m) \rtimes \mathbb{R}$ and is called the *group-measure von Neumann algebra* determined by $(X, \mathbb{R})$ and m or the *von Neumann algebra crossed product* determined by $\mathbb{R}$ acting on $L^\infty(X)$. If m is ergodic, then it is well known that $L^\infty(X) \rtimes \mathbb{R}$ is a II_∞-factor. While we do not want to present all of the details of the proof of this assertion here, certain aspects of the proof will be useful.

Observe that if we define τ on $C_c(X\times\mathbb{R})$ by the formula

$$\tau(a)=\int_X a(x,0)\,dm(x), \qquad a\in C_c(X\times\mathbb{R}),$$

then τ is a trace on $C_c(X\times\mathbb{R})$ and $C_c(X\times\mathbb{R})$ becomes a Hilbert algebra under the inner product $(a,b)=\tau(b^* * a)=\int_X\int_{\mathbb{R}} a(x,t)\,\overline{b(x,t)}\,dx\,dt$. The Hilbert space completion of $C_c(X\times\mathbb{R})$ in this inner product is $L^2(X\times\mathbb{R})$, of course, and we see at once that $L^\infty(m)\rtimes\mathbb{R}$ is the left von Neumann algebra of this Hilbert algebra. In particular, τ extends to be a faithful normal semifinite trace on $L^\infty(m)\rtimes\mathbb{R}$. We note too, that the achieved Hilbert algebra of $C_c(X\times\mathbb{R})$ is just $\mathfrak{K}_2(L^\infty(m)\rtimes\mathbb{R})$ and that an operator $K\in L^\infty(m)\rtimes\mathbb{R}$ belongs to $\mathfrak{K}_2(L^\infty(m)\rtimes\mathbb{R})$ if and only if there is a function $k\in L^2(X\times\mathbb{R})$ such that $(K\xi)(x,t)=\int_{\mathbb{R}} k(x,s)\,\xi(x+s,t-s)\,ds$, $\xi\in L^2(X\times\mathbb{R})$. The $\mathfrak{K}_2$-norm of K is then the L^2-norm of k. Since $\mathfrak{K}_1(L^\infty(m)\rtimes\mathbb{R})=(\mathfrak{K}_2(L^\infty(m)\rtimes\mathbb{R}))^2\subseteq\mathfrak{K}_2(L^\infty(m)\rtimes\mathbb{R})$, (relative) trace class elements of $L^\infty(m)\rtimes\mathbb{R}$ are represented by functions too, and their traces are calculated according to the same formula defining τ. In particular, we see that $\operatorname{Ind} m(C_c(X\times\mathbb{R}))\subseteq\mathfrak{K}_1(L^\infty(m)\rtimes\mathbb{R})$ and so $\operatorname{Ind} m(C_c(X)\times\mathbb{R})\subseteq\mathfrak{K}_\infty(L^\infty(m)\rtimes\mathbb{R})$. The last inclusion is usually proper because $C(X)\rtimes\mathbb{R}$ is separable, since X is, while $\mathfrak{K}_\infty(L^\infty(m)\rtimes\mathbb{R})$ is separable only when m is the sum of a countable number of measures each of which is concentrated of an orbit and is equivalent there to the transplant of Lebesgue measure on $\mathbb{R}$. Strictly speaking, this is ruled out by our assumption that m is invariant *and* finite; so in our setting, the inclusion is always proper. However, $L^\infty(m)\rtimes\mathbb{R}$ makes sense for possibly infinite, quasi-invariant measures, and for these, one has to consider the possibility that $\mathfrak{K}_\infty(L^\infty(m)\rtimes\mathbb{R})$ might be separable. But still, this happens only onder the circumstances just described.

Define V on $L^2(X\times\mathbb{R})$ by the formula $(V\xi)(x,t)=\xi(x+t,-t)$. Thus except for the complex conjugation, on $C_c(X\times\mathbb{R})$, V is just the map $a\to a^*$. Of course, V is a unitary operator that conjugates $L^\infty(m)\rtimes\mathbb{R}$ onto the right von Neumann algebra associated with the Hilbert algebra $C_c(X\times\mathbb{R})$; i.e., $V(L^\infty(m)\rtimes\mathbb{R})\,V^*=(L^\infty(m)\rtimes\mathbb{R})'$. In particular, note that $V\sigma^m(\varphi)\,V^*\xi(x,s)=\varphi(x+s)\,\xi(x,s)$, $\varphi\in L^\infty(m)$, $\xi\in L^2(X\times\mathbb{R})$, and that $VU_t^m(\varphi)\,V^*\xi(x,s)=\xi(x,s+t)$, $\xi\in L^2(X\times\mathbb{R})$. It follows that if we view $L^2(X\times\mathbb{R})$ as $L^2(X)\otimes L^2(\mathbb{R})$, then $\{U_t^m\}_{t\in\mathbb{R}}$ is diagonalized by $V^*(I\otimes\mathfrak{F})\,V$, where $\mathfrak{F}$ is the Fourier–Plancherel transform on $L^2(\mathbb{R})$. It follows that if E is the spectral measure of $\{U_t^m\}_{t\in\mathbb{R}}$, then for any Borel set M of finite Lebesgue measure $(E(M)\xi)(x,t)=\int_{\mathbb{R}}\hat{1}_M(s)\,\xi(x+s,t-s)\,ds$ where $\hat{1}_M$ is the Fourier transform of the indicator function of M, 1_M. That is, $E(M)$ is represented by the kernel $k(x,t)=\hat{1}_M(t)$. Thus,

$\tau(E(M)) = \int k(x, 0)\, dm(x) = \hat{1}_M(0) =$ the Lebesgue measure of M. Hence it is clear that when restricted to the projections in $L^\infty(m) \rtimes \mathbb{R}$, τ takes on all values between 0 and ∞.

17.4. LEMMA. *If $L^2(X \times \mathbb{R})$ is viewed as the direct integral $\int_X^\oplus \mathscr{H}_x\, dm(x)$, where $\{\mathscr{H}_x\}_{x \in X}$ is the constant field $\mathscr{H}_x \equiv L^2(\mathbb{R})$, then*

$$V \operatorname{Ind} m(\cdot)\, V^* = \int_X^\otimes \operatorname{Ind} \delta_x(\cdot)\, dx(x).$$

Proof. This is evident from the formulas in Section 17.3.

18.1. As in the introduction, $H^2(\mathbb{R})$ will denote the usual Hardy space on the line and P will denote the projection of $L^2(\mathbb{R})$ onto $H^2(\mathbb{R})$. For each $\varphi \in C(X)$ and $x \in X$, we obtain a Toeplitz operator T_φ^x by the formula

$$T_\varphi^x = P\sigma^x(\varphi) \,|\, H^2(\mathbb{R}).$$

The C^*-algebra generated by $\{T_\varphi^x \,|\, \varphi \in C(X)\}$ will be denoted by $\mathfrak{T}_x$.

18.2. On $L^2(X \times \mathbb{R})$, we define the (transferred) Hilbert transform H^m by the principal value integral

$$(H^m\xi)(x, t) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1}{s}\, \xi(x+s, t-s)\, ds,$$

$\xi \in L^2(X \times \mathbb{R})$; i.e., formally, $H^m = (1/\pi i) \int_{-\infty}^{\infty} (1/s)\, U_s^m\, ds$. If $L^2(X \times \mathbb{R})$ is viewed either as $L^2(X) \otimes L^2(\mathbb{R})$ or as $\int_X^\oplus L^2(\mathbb{R})\, dm(x)$ (we now drop the heavy terminology and notation, and write $L^2(\mathbb{R})$ for $\mathscr{H}_x$), then we may write $VH^mV^* = I \otimes H = \int_X H^x\, dm(x)$, where H and H^x, $x \in X$, all denote the Hilbert transform on $L^2(\mathbb{R})$. Thus H^m is unitary and satifies the equation $(H^m)^2 = I$. The operator $P^m \equiv (I + H^m)/2$, then, is a projection and its range will be denoted $H^2(X \times \mathbb{R})$. Evidently, $H^2(X \times \mathbb{R}) = V(L^2(X) \otimes H^2(\mathbb{R})) = V(\int_X^\oplus H^2(\mathbb{R})\, dm(x))$, and $P^m = E([0, \infty))$, where, as in Section 17.3, E is the spectral measure of $\{U_t^m\}_{t \in \partial\mathbb{R}}$.

18.3. For $\varphi \in C(X)$, we define T_φ^m on $H^2(X \times \mathbb{R})$ by the formula

$$T_\varphi^m = P^m\sigma^m(\varphi) \,|\, H^2(X \times \mathbb{R}).$$

Evidently, for each $\varphi \in C(X)$, $\{T_\varphi^x\}_{x \in X}$ is a bounded measurable field of operators, in fact, it is a strongly continuous field in the sense that for each

$\xi \in H^2(\mathbb{R})$, the function $x \to T_\varphi^x \xi$ from X to $H^2(\mathbb{R})$ is continuous, and from Lemma 17.4 it is immediate that

$$T_\varphi^m = V\left(\int_X^\oplus T_\varphi^x \, dm(x)\right) V^* \Big| H^2(X \times \mathbb{R}). \tag{18.1}$$

The C^*-algebra generated by $\{T_\varphi^m \mid \varphi \in C(X)\}$ will be denoted by $\mathfrak{T}_m$.

19. THEOREM. *Suppose that $(X, \mathbb{R})$ is minimal, and for each $x \in X$, let ρ_x be the linear map from $\{T_\varphi^x \mid \varphi \in C(X)\}$ into $\mathfrak{T}_m$ defined by the equation*

$$\rho_x(T_\varphi^x) = T_\varphi^m, \qquad \varphi \in C(X).$$

Then ρ_x extends to a C^*-isomorphism from $\mathfrak{T}_x$ onto $\mathfrak{T}_m$. In particular, for all x and y, $\mathfrak{T}_x$ and *$\mathfrak{T}_y$ are isomorphic.*

Proof. Fix a finite collection of functions, φ_{jk}, and form the operators

$$A_m = \sum_j \prod_k T_{\varphi_{jk}}^m \qquad \text{on } H^2(X \times \mathbb{R}),$$

and

$$A_x = \sum_j \prod_k T_{\varphi_{jk}}^x \qquad \text{on } H^2(\mathbb{R}), \qquad x \in X.$$

It suffices to prove that $\|A_m\| = \|A_x\|$ for all x. Now $\{A_x\}_{x \in X}$ is a strongly continuous field of operators and from Lemma 17.4 and Eq. (18.1), we see that

$$A_m = V\left(\int_X^\oplus A_x \, dm(x)\right) V^* \Big| H^2(X \times \mathbb{R}).$$

Thus, $\|A_m\| = \sup_{x \in X} \|A_x\|$. Hence, it is enough to show that $\|A_x\| = \|A_y\|$ for all x and y. For this, first note that if $t \in \mathbb{R}$, $A_{x+t} = U_t A_x U_t^*$, where $\{U_t\}_{t \in \mathbb{R}}$ is defined on $H^2(\mathbb{R})$ by $(U_t \xi)(s) = \xi(s+t)$. Thus $\|A_{x+t}\| = \|A_x\|$ for all t. By the minimality of $(X, \mathbb{R})$ we may choose a sequence $\{t_n\}_{n=1}^\infty$ in $\mathbb{R}$ so that $x + t_n \to y$. Then $A_{x+t_n} \to A_y$ strongly, and we conclude that $\|A_y\| \leqslant \underline{\lim} \|A_{x+t_n}\| = \|A_x\|$. Reversing the roles of x and y, we find that $\|A_x\| = \|A_y\|$, and this completes the proof.

20. *Digression.* In one sense we have now associated with each (minimal) flow $(X, \mathbb{R})$ a C^*-algebra which might well qualify as *the* Toeplitz C^*-algebra of $(X, \mathbb{R})$; namely, any of the $\mathfrak{T}_x$'s or $\mathfrak{T}_m$. We shall write $\mathfrak{T}_0(X, \mathbb{R})$ for any of these algebras. However, there are two things which we find unsatisfactory about this. First of all, there are other algebras that might

well qualify as candidates for being called *the* Toeplitz algebra on X. As an example, consider the space $H^2(m)$ which is defined to be the collection of those functions $\xi \in L^2(m)$ such that for almost all x, the function of t, $\xi(x+t)$, lies in the Hardy space $H^2(1/(1+t^2))$ consisting of those functions that lie in the closure of $H^\infty(\mathbb{R})$ in the L^2-space based on the measure $dt/(1+t^2)$. If $\tilde{P}$ denotes the projection of $L^2(m)$ onto $H^2(m)$, if, for $\varphi \in C(X)$, we define $\tilde{T}_\varphi$ on $H^2(m)$ by the equation

$$\tilde{T}_\varphi \xi = \tilde{P}\varphi\xi, \qquad \xi \in H^2(m),$$

and if we let $\tilde{\mathfrak{T}}(X, \mathbb{R})$ denote the C^*-algebra generated by $\{\tilde{T}_\varphi \mid \varphi \in C(X)\}$, we obtain a C^*-algebra which is a good candidate to be called the Toeplitz C^*-algebra of X. For any $x \in X$ (assuming $(X, \mathbb{R})$ is minimal), the map $T^x_\varphi \to \tilde{T}_\varphi$, $\varphi \in C(X)$, is a self-adjoint, isometric, linear map between $\{T^x_\varphi \mid \varphi \in C(X)\}$ and $\{\tilde{T}_\varphi \mid \varphi \in C(X)\}$, but at this time we are unable to decide if it extends to a $*$-isomorphism between $\mathfrak{T}_x$ and $\tilde{\mathfrak{T}}(X, \mathbb{R})$. We note, however, that if the flow is almost periodic, then the extension is possible. Indeed, all reasonable definitions of the Toeplitz C^*-algebra on X coincide. The reason for this is that they all are generated by isometric representations of the "positive half" of the subgroup of $\mathbb{R}$ that is dual to X and these, in turn, where shown by Douglas [D] to generate isomorphic C^*-algebras. However, in general, as we shall see, $\mathfrak{T}_x$ need not contain any nonunitary isometries at all.

The second dissatisfying aspect of our definition of $\mathfrak{T}_0(X, \mathbb{R})$ is that it is not intrinsic. We first produced a bunch of spatially defined C^*-algebras, and then showed that they are all naturally isomorphic. There is no concretely defined object, with its own ontological status, as it were, whose representations include $\mathfrak{T}_x$ and $\mathfrak{T}_m$. We have in mind something like the definition of $C(X) \rtimes \mathbb{R}$. It is a completion of $C_c(X \times \mathbb{R})$ that exists independently of any particular representation.

However, if one ponders $C(X) \rtimes \mathbb{R}$ a little, one can discover an intrinsic Toeplitz C^*-algebra on X. Consider the double dual of $C(X) \rtimes \mathbb{R}$. This is a huge von Neumann algebra, which we denote by $W^*(X, \mathbb{R})$, acting on a nonseparable Hilbert space. The C^*-algebra $C(X) \rtimes \mathbb{R}$ is imbedded isometrically in $W^*(X, \mathbb{R})$ and the image is weakly dense in $W^*(X, \mathbb{R})$. As we noted in Section 17.2, this representation of $C(X) \rtimes \mathbb{R}$ must begiven by a covariant representation (σ^u, V^u). (The superscript u is for "universal.") Let E^u be the spectral measure for $\{V^u_t\}_{t \in \mathbb{R}}$. Then, of course, the values of E^u lie in $W^*(X, \mathbb{R})$. We define $SI(X, \mathbb{R})$ to be the C^*-subalgebra of $W^*(X, \mathbb{R})$ generated by $\sigma^u(C(X))$ and $E^u([0, \infty))$ and we call $SI(X, \mathbb{R})$ the *C^*-algebra of singular integral operators* based on the flow $(X, \mathbb{R})$. The reason for the terminology is that if X were the one-point compactification of $\mathbb{R}$, with $\mathbb{R}$ acting in the usual fashion, then $SI(X, \mathbb{R})$ would coincide with the familiar C^*-algebra on $L^2(\mathbb{R})$ generated by singular integral operators with continuous symbols each tending to a limit at ∞. Note, that while $W^*(X, \mathbb{R})$ is highly nonseparable and pathological, $SI(X, \mathbb{R})$ is much smaller and significantly better behaved. In particular, $SI(X, \mathbb{R})$ is separable since X is. One candidate for the Toeplitz C^*-algebra on the flow, $\mathfrak{T}(X, \mathbb{R})$, is simply the "corner"

$$E^u([0, \infty))\, SI(X)\, E^u([0, \infty)).$$

Note, again, that if X is the one-point compactification of $\mathbb{R}$, then $\mathfrak{T}(X, \mathbb{R})$ is as it should be.

There are a couple of nice features of $\mathfrak{T}(X, \mathbb{R})$. First, it is intrinsically defined; i.e., its definition does not depend on aıny particular representation. (One does not have to know that $W^*(X, \mathbb{R})$ is represented anywhere to carry out the construction of $\mathfrak{T}(X, \mathbb{R})$.) As a result, one does not have to make any special hypotheses on $(X, \mathbb{R})$, like minimality, to study $\mathfrak{T}(X, \mathbb{R})$ efficiently. Second, every representation of $C(X) \rtimes \mathbb{R}$, π, say, gives rise to a representation of $\mathfrak{T}(X, \mathbb{R})$. One extends π to $W^*(X, \mathbb{R})$ in the usual way, and then restricts the extension to obtain a representation of $\mathfrak{T}(X, \mathbb{R})$. In particular, we see that the algebras $\mathfrak{T}_x$, $x \in X$, $\mathfrak{T}_m$, and $\tilde{\mathfrak{T}}(X, \mathbb{R})$ are representations of $\mathfrak{T}(X, \mathbb{R})$. In fact, it seems that the relation between representations of $\mathfrak{T}(X, \mathbb{R})$ and $C(X) \rtimes \mathbb{R}$ is almost reciprocal. We are able to prove in many instances that a representation of $\mathfrak{T}(X, \mathbb{R})$ that does not annihilate the commutator ideal of $\mathfrak{T}(X, \mathbb{R})$ must come from a representation of $C(X) \rtimes \mathbb{R}$ in the fashion just described.

The one disadvantage of the definition of $\mathfrak{T}(X, \mathbb{R})$, at least so far, is that we do not have any idea about the ideal structure of $\mathfrak{T}(X, \mathbb{R})$, even if $(X, \mathbb{R})$ is minimal. In particular, we do not know if any of the representations of $\mathfrak{T}(X, \mathbb{R})$, such as $\mathfrak{T}_x$, $\mathfrak{T}_m$, or $\tilde{\mathfrak{T}}(X, \mathbb{R})$, is faithful. In fact, we tried very hard to show that $\mathfrak{T}_x$ and $\tilde{\mathfrak{T}}(X, \mathbb{R})$ are isomorphic by showing that they both are faithful representations of $\mathfrak{T}(X, \mathbb{R})$. In any event, this is the primary reason why we concentrate our attention here on $\mathfrak{T}_0(X, \mathbb{R})$ rather than on $\mathfrak{T}(X, \mathbb{R})$.

We speculate that when $(X, \mathbb{R})$ is minimal, then the commutator ideals of $\mathfrak{T}(X, \mathbb{R})$ and of $SI(X, \mathbb{R})$ are simple. This is known to be true in the almost periodic case [D], but we do not know what happens in general. The commutator ideal of $SI(X, \mathbb{R})$ seems to be related to $C(X) \rtimes \mathbb{R}$ and one might expect that in fact they coincide. (If they do, then, the reciprocity between the representations of $C(X) \rtimes \mathbb{R}$ and the representations of $\mathfrak{T}(X, \mathbb{R})$ not annihilating the commutator ideal would almost be assured.) However, they are different. One reason in the almost periodic case is that when the flow is almost periodic, the commutator ideal of $SI(X, \mathbb{R})$ contains certain nontrivial spectral projections for $\{V_t^u\}_{t \in \mathbb{R}}$. On the other hand, $C(X) \rtimes \mathbb{R}$

contains no spectral projections of $\{V_t^u\}_{t\in\mathbb{R}}$ other than 0 and I. Jerry Kaminker has informed us that Ian Putnam has shown that in general K_1 of the commutator ideal in $SI(X, \mathbb{R})$ is *not* equal to $K_1(C(X)\rtimes\mathbb{R})$. Thus the commutator ideal of $SI(X, \mathbb{R})$ is not even isomorphic to $C(X)\rtimes\mathbb{R}$. There is an additional importance of this fact for our theory. It shows that the algebras we are studying, while related to Connes's theory of C^*-algebras and foliations, are slightly different from the ones he studies. To be more specific, suppose that X is a manifold and that the flow is determined by a differential equation. Then we can think of the flow as a foliation. In fact, $C(X)\rtimes\mathbb{R}$ is the C^*-algebra of this foliation. One might then expect that $SI(X, \mathbb{R})$ coincides with his algebra, $\Psi_0(X, \mathbb{R})$, of order zero pseudo-differential operators along the foliation, i.e., along the flow. However, it does not because the commutator ideal of $\Psi_0(X, \mathbb{R})$ *is* $C(X)\rtimes\mathbb{R}$ while the commutator ideal of $SI(X, \mathbb{R})$ is not. The difference, we think, is due to the fact that $\Psi_0(X, \mathbb{R})$ is built up from singular integrals whose singularities lie only at finite points of $\mathbb{R}$, while $SI(X, \mathbb{R})$ "contains" the Hilbert transform H, and H has singularities *both at* 0 *and at* ∞.

21. We set $\mathfrak{N} = P^m(L^\infty(m)\rtimes\mathbb{R})\, P^m \mid H^2(X\times\mathbb{R})$. Since P^m is in $L^\infty(m)\rtimes\mathbb{R}$, $\mathfrak{N}$ is just a corner of $L^\infty(m)\rtimes\mathbb{R}$ and the restriction of τ to $\mathfrak{N}$ is a faithful normal semifinite trace on $\mathfrak{N}$. We will not distinguish notationally between τ and its restriction to $\mathfrak{N}$.

21.1. *Remark.* The following two facts are worth noting. However, since they are not necessary for the results of this paper and since their proofs would take us too far afield, we will not prove them here. The first is a consequence of some results in [M4], while the proof of the second requires somewhat delicate arguments from the theory of spectral synthesis.

(i) If x is not a periodic point, then $\mathfrak{T}_x$ is irreducible.

(ii) The von Neumann algebra generated by $\mathfrak{T}_m$ is $\mathfrak{N}$.

22. In this section we are concerned with certain commutators of operators in $L^\infty(m)\rtimes\mathbb{R}$ and in $\mathfrak{T}_m$. To lighten the notation, we omit the superscript m in our calculations involving σ^m, P^m, H^m, etc.

22.1. Lemma. *If* $\varphi\in C^1(X)$, *then the commutator* $[H, \sigma(\varphi)]$ *lies in* $\mathfrak{K}_2(L^\infty(m)\rtimes\mathbb{R})$, *and*

$$\tau([H, \sigma(\varphi)]^*[H, \sigma(\varphi)]) = \frac{1}{\pi^2}\int_X\int_{\mathbb{R}}\left|\frac{1}{s}(\varphi(x+s)-\varphi(x))\right|^2 ds\, dm(x).$$

Proof. A straightforward calculation shows that the commutator $[H, \sigma(\varphi)]$ is given by the kernel

$$k(x, s) = \frac{1}{\pi s}(\varphi(x+s) - \varphi(x)).$$

Since $\varphi \in C^1(X)$ by assumption, $k \in L^2(X \times \mathbb{R})$. So, as we noted in Section 17.3, $[H, \sigma(\varphi)] \in \mathfrak{K}_2(L^\infty(m) \rtimes \mathbb{R})$, and the $\mathfrak{K}_2(L^\infty(m) \rtimes \mathbb{R})$-norm of $[H, \sigma(\varphi)]$ is the L^2-norm of k. This proves the asserted equation.

23.1. THEOREM. *For $\varphi, \psi \in C^1(X)$, $[T_\varphi, T_\psi]$ lies in $\mathfrak{K}_1(\mathfrak{N})$ and*

$$\tau([T_\varphi, T_\psi]) = \frac{-1}{2\pi i}\int_X \varphi'(x)\,\psi(x)\,dm(x). \tag{23.1}$$

Proof. Decompose $L^2(X \times \mathbb{R})$ as $H^2(X \times \mathbb{R}) \oplus H^2(X \times \mathbb{R})^\perp =$ ran $P \oplus$ ran Q. Then, for $\varphi \in C(X)$, $\sigma(\varphi)$ has the matrix representation

$$\sigma(\varphi) = \begin{pmatrix} T_\varphi & H_\varphi \\ H_{\bar\varphi}^* & S_\varphi \end{pmatrix},$$

where $H_\varphi = P\sigma(\varphi)Q = P[P, \sigma(\varphi))] = P[H, \sigma(\varphi)]/2$ is the Hankel operator determined by φ, and $S_\varphi = Q\sigma(\varphi)Q$. If $\varphi \in C^1(X)$, then $H_\varphi \in \mathfrak{K}_2(L^\infty(m) \rtimes \mathbb{R})$ by Lemma 22.1. It follows at once that $[T_\psi, T_\varphi] = H_\varphi H_{\bar\psi}^* - H_\psi H_{\bar\varphi}^* \in \mathfrak{K}_1(L^\infty(m) \rtimes \mathbb{R})$, if $\varphi, \psi \in C^1(X)$, and that $\tau(H_{\bar\psi}^* H_\varphi - H_{\bar\varphi}^* H_\psi) = \tau(H_\varphi H_{\bar\psi}^* - H_\psi H_{\bar\varphi}^*) = \tau([T_\psi, T_\varphi])$. Consequently, since $P = (I+H)/2$, we see that

$$\begin{aligned} \tau([T_\psi, T_\varphi]) &= \tfrac{1}{2}\tau([T_\psi, T_\varphi] + H_\varphi H_{\bar\psi}^* - H_\psi H_{\bar\varphi}^*) \\ &= \tfrac{1}{4}\,\tau(P(\sigma(\psi)\,H\sigma(\varphi) - \sigma(\varphi)\,H\sigma(\psi))P \\ &\quad + Q(\sigma(\psi)\,H\sigma(\varphi) - \sigma(\varphi)\,H\sigma(\psi))Q). \end{aligned}$$

By what was just shown, $\sigma(\psi)\,H\sigma(\varphi) - \sigma(\varphi)\,H\sigma(\psi)$ lies in $\mathfrak{K}_2(L^\infty(m) \rtimes \mathbb{R})$. If it were the case that $\sigma(\psi)\,H\sigma(\varphi) - \sigma(\varphi)\,H\sigma(\psi)$ is in $\mathfrak{K}_1(L^\infty(m) \rtimes \mathbb{R})$, then the last expression would collapse to $\frac{1}{4}\tau(\sigma(\psi)\,H\sigma(\varphi) - \sigma(\varphi)\,H\sigma(\psi))$. However, we do not know if this expression is in $\mathfrak{K}_1(L^\infty(m) \rtimes \mathbb{R})$. Nevertheless, we can "sum" its trace as follows. Let $\{f_n\}_{n=1}^\infty$ be a sequence of compactly supported continuous functions on $\mathbb{R}$ that forms an approximate identity for $L^1(\mathbb{R})$, and view each f_n as a function in $C_c(X \times \mathbb{R})$ that is constant on X. If we set $F_n = \operatorname{Ind} m(f_n)$, then as we noted in Section 17.4, each F_n lies in $\mathfrak{K}_1(L^\infty(m) \rtimes \mathbb{R})$ and it is easy to see that $\{F_n\}_{n=1}^\infty$ converges strongly to I. To calculate the trace of $F_n(\sigma(\psi)\,H\sigma(\varphi) - \sigma(\varphi)\,H\sigma(\psi))$ simply represent it as an inner product in $\mathfrak{K}_2(L^\infty(m) \rtimes \mathbb{R})$ and then use the kernels representing F_n and

$(\sigma(\psi)\, H\sigma(\varphi) - \sigma(\varphi)\, H\sigma(\psi))$ to calculate this inner product as one in $L^2(X \times \mathbb{R})$. The result is that

$$\begin{aligned}
&\tau(F_n(\sigma(\psi)\, H\sigma(\varphi) - \sigma(\varphi)\, H\sigma(\psi))) \\
&\quad = \frac{1}{\pi i} \int_X \int_{\mathbb{R}} f_n(s) \left(\frac{-1}{s}\right) [\psi(x+s)\, \varphi(x) \\
&\qquad - \varphi(x+s)\, \psi(x)]\, ds\, dm(x). \qquad (23.2)
\end{aligned}$$

If we add and subtract $\psi(x)\,\varphi(x)$ inside the backets, then because $\{f_n\}_{n \in \mathbb{R}}^{\infty}$ is an approximate identity for $L^1(\mathbb{R})$, we see that the limit of the integrals in (23.2) is

$$\frac{1}{\pi i} \int_X \varphi'(x)\, \psi(x) - \varphi(x)\, \psi'(x)\, dm(x).$$

If we evaluate this integral using the individual ergodic theorem, we obtain

$$\lim_{T \to \infty} \frac{1}{\pi i} \frac{1}{2T} \int_{-T}^{T} \varphi'(x+t)\, \psi(x+t) - \varphi(x+t)\, \psi'(x+t)\, dt \qquad \text{a.e.}$$

and if we integrate by parts, we obtain

$$\lim_{T \to \infty} \frac{2}{\pi i} \frac{1}{2T} \int_{-T}^{1} \varphi'(x+t)\, \psi(x+t)\, dt \qquad \text{a.e.}$$

Applying the individual ergodic theorem, again, we conclude that the limit of the integrals in Eq. (23.2) is

$$\frac{2}{\pi i} \int_X \varphi'(x)\, \psi(x)\, dm(x).$$

On the other hand, since F_n commutes with P and Q, we see that

$$\begin{aligned}
\tau([T_\psi, T_\varphi]) &= \lim_{n \to \infty} \tau(F_n[T_\psi, T_\varphi]) \\
&= \frac{1}{2} \lim_{n \to \infty} \tau(F_n P(\sigma(\psi)\, P\sigma(\varphi) - \sigma(\varphi)\, P\sigma(\psi))P \\
&\qquad + F_n Q(\sigma(\psi)\, P\sigma(\varphi) - \sigma(\varphi)\, P\sigma(\psi))Q) \\
&= \frac{1}{4} \lim_{n \to \infty} \tau(F_n(\sigma(\psi)\, H\sigma(\varphi) - \sigma(\varphi)\, H\sigma(\psi))) \\
&= \frac{1}{2\pi i} \int_X \varphi'(x)\, \psi(x)\, dm(x).
\end{aligned}$$

This completes the proof.

24. Let $\mathfrak{C}_m$ be the closed, two-sided ideal in $\mathfrak{T}_m$ generated by $\{T^m_\varphi T^m_\psi - T^m_{\varphi\psi} \mid \varphi, \psi \in C(X)\}$ and for $x \in X$, let $\mathfrak{C}_x$ be the closed, two-sided ideal in $\mathfrak{T}_x$ generated by $\{T^x_\varphi T^x_\psi - T^x_{\varphi\psi} \mid \varphi, \psi \in C(X)\}$. It is clear that, in the notation of Theorem 19, $\rho_x(\mathfrak{C}_x) = \mathfrak{C}_m$, assuming $(X, \mathbb{R})$ is minimal. It is also clear that $\mathfrak{C}_x$ and $\mathfrak{C}_m$ contain the commutator ideals of $\mathfrak{T}_x$ and $\mathfrak{T}_m$, respectively. However, we are unable to decide, in general, if these ideals coincide with the commutator ideals. The best we can say is the following.

24.1. LEMMA. *If $A(X, \mathbb{R})$ is a Dirichlet algebra on X (so, in particular, if $(X, \mathbb{R})$ is strictly ergodic), then $\mathfrak{C}_x$ coincides with the commutator ideal of $\mathfrak{T}_x$ and $\mathfrak{C}_m$ coincides with the commutator ideal of $\mathfrak{T}_m$.*

Proof. We write T_φ for T^x_φ or T^m_φ. Since functions of the form $a + b$, $a \in A(X, \mathbb{R})$, $\bar{b} \in A(X, \mathbb{R})$, are dense in $C(X)$, it suffices to express $T_{(a+b)} T_\psi - T_{(a+b)\psi}$ as a commutator, where $\psi \in C(X)$ is arbitrary and $a, \bar{b} \in A(X, \mathbb{R})$. Since $T_{b\psi a} = T_b T_\psi T_a$ in this case, we have $T_{a+b} T_\psi - T_{(a+b)\psi} = T_a T_\psi + T_b T_\psi - T_{b\psi} - T_{\psi a} = T_a T_\psi + T_b T_\psi - T_b T_\psi - T_\psi T_a = [T_a, T_\psi]$.

24.2. LEMMA. *Suppose that $(X, \mathbb{R})$ is minimal. Then for each x, $\mathfrak{T}_x/\mathfrak{C}_x$ and $\mathfrak{T}_m/\mathfrak{C}_m$ are isomorphic, and they all are isomorphic to $C(X)$.*

Proof. The first assertion is clear from Theorem 19. To prove the second, it suffices to show that $\|\varphi\|_\infty = \|T^x_\varphi + \mathfrak{C}_x\|$ for any $x \in X$ and $\varphi \in C(X)$. The inequality $\|T^x_\varphi + \mathfrak{C}_x\| \leqslant \|\varphi\|_\infty$ is obvious, and the reverse inequality follows easily from this assertion:

Assertion. For each finite collection of functions $\varphi_{jk} \in C(X)$, and for any $\xi, \eta \in H^2(\mathbb{R})$,

$$\lim_{\lambda \to \infty} \left(\sum_j \prod_k T^x_{\varphi_{jk}} W_\lambda \xi, W_\lambda \eta \right)$$
$$= \left(\sum_j \prod_k \sigma^x(\varphi_{jk}) \xi, \eta \right) = (T^x_\psi \xi, \eta), \tag{24.1}$$

where $(W_\lambda \xi)(t) = e^{i\lambda t} \xi(t)$ and $\psi = \sum_j \prod_k \varphi_{jk}$.

To prove the assertion, it suffices to prove that if $\xi \in L^2(\mathbb{R})$ and if $\varepsilon > 0$ is given, then there is an $L > 0$ such that $\|PW_\lambda \xi - W_\lambda \xi x\|_2 < \varepsilon$ for all $\lambda > L$. However, by Plancherel's theorem, $\|PW_\lambda \xi - W_\lambda \xi\|_2 = \|1_{(-\infty, -\lambda}(\mathfrak{F}\xi)\|_2$ where $\mathfrak{F}\xi$ is the Fourier–Plancherel transform of ξ. Since $\mathfrak{F}\xi \in L^2(\mathbb{R})$, this last expression goes to zero as $\lambda \to \infty$.

To see how the inequality $\|T^x_\varphi + \mathfrak{C}_x\| \leqslant \|\varphi\|_\infty$ follows from the assertion, observe that $\|T^x_\varphi + \mathfrak{C}_x\| = \inf \|T^x_\varphi - A\|$, where A runs over all sums of products of operators of the form $T^x_{\psi_1} T^x_{\psi_2} - T^x_{\psi_1 \psi_2}$, $\psi_1, \psi_2 \in C(X)$.

The assertion implies that for all $\xi, \eta \in H^2$, $\lim_{\lambda \to \infty}(AW_\lambda \xi, W_\lambda \eta) = 0$ for each such A. But then we have, for all such A, $\|T_\varphi^{Ax} + A\| = \sup\{|((T_\varphi^x + A)\xi, \eta)| \mid \|\xi\|, \|\eta\| = 1\} \geqslant \sup\{|((T_\varphi^x + A) W_\lambda \xi, W_\lambda \eta)| \mid \|\xi\|, \|\eta\| = 1\}$ for all $\lambda > 0 \geqslant \sup\{\overline{\lim}_\lambda |((T_\varphi^x + A) W_\lambda \xi, W_\lambda \eta)| \mid \|\xi\|, \|\eta\| = 1\} = \sup\{|(T_\varphi^x \xi, \eta)| \mid \|\xi\|, \|\eta\| = 1\} = \|\varphi\|_\infty$. This completes the proof.

24.3. Lemma. *Assume $(X, \mathbb{R})$ is minimal. Then, for $\varphi \in C(X)$, $T_\varphi^m \in \mathfrak{K}_\infty(\mathfrak{N})$ if and only if $\varphi \equiv 0$.*

Proof. Observe first that $\mathfrak{T}_m \cap \mathfrak{K}_\infty(\mathfrak{N})$ is an ideal in $\mathfrak{T}_m$ that contains $\mathfrak{C}_m$ by the argument in the first paragraph of the proof of Theorem 23.1. We have for all $\varphi, \psi \in C(X)$, $T_\varphi^m T_\psi^m - T_{\varphi\psi}^m = (P^m \sigma^m(\varphi) P^m \sigma^m(\psi) P^m - P^m \sigma^m(\varphi) \sigma^m(\psi) P^m) \mid H^2(X \times \mathbb{R}) = -(P^m \sigma^m(\varphi) Q^m \sigma^m(\psi) P^m \mid H^2(X \times \mathbb{R})$. But also from the proof we know that $P^m \sigma^m(\varphi) Q^m$ is in $\mathfrak{K}_2(L^\infty(m) \rtimes \mathbb{R})$ when $\varphi \in C^1(X)$. Thus, it follows by approximation that $(P^m \sigma^m(\varphi) Q^m \sigma^m(\psi) P^m) | H^2(X \times \mathbb{R})$ is compact for $\varphi, \psi \in C(X)$. Suppose, now, that $T_\varphi^m \in \mathfrak{K}_\infty(\mathfrak{N})$. Then since P^m is a spectral projection for $\{U_t^m\}_{t \in \mathbb{R}}$, we have $U_t^m T_\varphi^m (U_t^m)^* = T_{\varphi_t}^m$, where $\varphi_t(x) = \varphi(x + t)$, and so $T_{\varphi_t}^m$ lies in $\mathfrak{K}_\infty(\mathfrak{N})$ for all t. It follows that $(\mathfrak{T}_m \cap \mathfrak{K}_\infty(\mathfrak{N}))/\mathfrak{C}_m$ is a translation invariant ideal in $\mathfrak{T}_m/\mathfrak{C}_m$. Since $\mathfrak{T}_m/\mathfrak{C}_m$ is isomorphic to $C(X)$ by Lemma 24.2 (note that the isomorphism is clearly equivariant), we conclude from the minimality of $(X, \mathbb{R})$, that the ideal is either zero or all of $\mathfrak{T}_m/\mathfrak{C}_m$. Since $I \notin \mathfrak{K}_\infty(\mathfrak{N})$, the ideal must be zero, and so there are no nonzero relatively compact Toeplitz operators.

If A is an operator in a II_∞ factor $\mathfrak{M}$, then we define the essential spectrum of A, $\sigma_{\text{ess}}(A)$, to be the spectrum of the image of A in the quotient algebra $\mathfrak{M}/\mathfrak{K}_\infty(\mathfrak{M})$. Equivalently, $\sigma_{\text{ess}}(A) = \{\lambda \in \mathbb{C} \mid A - \lambda$ is not Breuer–Fredholm$\}$. The preceding two lemmas now combine to prove the following theorem and corollary. No further details are necessary.

24.4. Theorem. *Assume that $(X, \mathbb{R})$ is minimal. Then for each $x \in X$, $\mathfrak{T}_x/\mathfrak{C}_x$ is isomorphic to $C(X)$ and for $\varphi \in C(X)$, $T_\varphi^m = \rho_x(T_\varphi^x)$ is Breuer–Fredholm in $\mathfrak{N}$ is and only if φ is invertible in $C(X)$.*

24.5. Corollary. *If $(X, \mathbb{R})$ is minimal, then for each $\varphi \in C(X)$, $\sigma_{\text{ess}}(T_\varphi^m) = \varphi(X)$, the range of φ.*

24.6. Corollary. *If $(X, \mathbb{R})$ is minimal, then*

$$\mathfrak{T}_m = \{T_\varphi^m + K \mid \varphi \in C(X), K \in \mathfrak{C}_m\}.$$

25. Our next objective is to compute the Breuer–Fredholm index of T_φ^m. Recall that this is given by the formula

$$\text{Index}(T_\varphi^m) = \tau(N(T_\varphi^m)) - \tau(N((T_\varphi^m)^*)),$$

where $N(T_\varphi^m)$ and $N((T_\varphi^m)^*)$ are the projections onto the kernels of T_φ^m and $(T_\varphi^m)^*$. We will calculate this index through the intervention of the Pincus principal function $g(\varphi; \cdot)$ associated with T_φ^m.

To this end, suppose that $\varphi = u + iv$ lies in $C^1(X)$. Then by Theorem 23.1, the commutator $[T_u^m, T_v^m]$ lies in $\mathfrak{K}_1(\mathfrak{N})$. Consequently, the smooth functional calculus that was introduced by Carey and Pincus in [CaP1] can be applied to the pair T_u^m, T_v^m. Let us recall, briefly, how this is done. Suppose that A and B are self-adjoint operators in a II_∞-factor $\mathfrak{M}$ and suppose that the commutator $[A, B]$ lies in $\mathfrak{K}_1(\mathfrak{M})$. Then for every function F, of two real-variables, that can be expressed locally as the Fourier–Stieltjes transform of a measure ω on $\mathbb{R}^2$ such that $\int\int (1+|s|)(1+|t|)\, d|\omega|(s, t) < \infty$, the iterated integral

$$\int \left(\int F(x, y)\, dE^A(x) \right) dE^B(y)$$

converges strongly and defines an operator in $\mathfrak{M}$, denoted $F(A, B)$. (Here, of course, E^A and E^B are the spectral measures of A and B.) Moreover, if F and G are two such functions, so is their pointwise product and $F(A, B)\, G(A, B) - (FG)(A, B)$ lies in $\mathfrak{K}_1(\mathfrak{M})$ [CaP1]. We are interested primarily in functions that lie in $C^\infty(\mathbb{R}^2)$ and we treat $C^\infty(\mathbb{R}^2)$ as a Lie algebra under the Poisson bracket $\{\cdot, \cdot\}$; i.e., for $F, G \in C^\infty(\mathbb{R}^2)$, $\{F, G\} = (\partial F/\partial x)(\partial G/\partial y) - (\partial F/\partial y)(\partial G/\partial x) = 2i(\bar{\partial} F\, \partial G - \partial F\, \bar{\partial} G)$. We write $F(T_\varphi^m)$ for $F(T_u^m, T_v^m)$ and we write $\tau_\varphi(F, G) = \tau([F(T_\varphi^m), G(T_\varphi^m)])$. Note that $\tau_\varphi(\cdot, \cdot)$ makes sense since $[T_u^m, T_v^m]$ and, hence $[F(T_\varphi^m), G(T_\varphi^m)]$, are in $\mathfrak{K}_1(\mathfrak{N})$. In the language of [Co4], τ_φ is a cyclic cocycle on $C^\infty(\mathbb{R}^2)$. Carey and Pincus [CaP1] show that τ_φ is a cyclic coboundary in the sense that there is a current $[g_\varphi]$ such that

$$\tau_\varphi(F, G) = \langle [g_\varphi], \{F, G\} \rangle.$$

In fact, this current is $(1/2\pi i)\, g(\varphi; z)\, d\mathfrak{L}^2((z)$, where $\mathfrak{L}^2$ denotes Lebesgue measure on $\mathbb{R}^2$ and $g(\varphi; z)$ is a uniquely determined function in $L^1(\mathbb{R}^2, \mathfrak{L}^2)$ called the Pincus principal function. It is supported in the disc $|z| \leqslant \|\varphi\|_\infty$ [CaP1].

25.1. THEOREM. *For $\varphi \in C^1(X)$, the principal function $g(\varphi; \cdot)$ is given by the formula*

$$g(\varphi; z) = \frac{-1}{2\pi i} \int_X \frac{\varphi'(x)}{\varphi(x) - z}\, dm(x).$$

Proof. For those z's for which the integral is finite, set $h(z) = (-1/2\pi i) \int_X (\varphi'(x)/(\varphi(x) - z))\, dm(x)$. Then by Fubini's theorem, h is defined almost everywhere, and is locally integrable on the plane.

Moreover, for $|z| > \|\varphi\|_\infty$, h is analytic and a calculation shows that for $n=0, 1, \ldots, h^{(n)}(\infty)=0$. Thus h vanishes outside the disc of radius $\|\varphi\|_\infty$ and so $h \in L^1(\mathbb{R}^2, \mathfrak{L}^2)$.

Let F be a polynomial in z and $\bar{z}$, and let f be a C^∞ function $\mathbb{R}^2$ that is identically 1 on the disc $|z| \leqslant \|\varphi\|_\infty$ and vanishes for $|z| \geqslant \|\varphi\|_\infty + 1$. Recalling the definition of $F(T_\varphi^m)$ just given and the properties of the functional calculus, we see that $(fF)(T_\varphi^m) = F(T_\psi^m)$, $(fF)(\varphi) = F(\varphi)$, and $F(T_\varphi^m) - T_{F(\varphi)}^m \in \mathfrak{K}_1(\mathfrak{N})$. So, by Theorem 23.1 we obtain the equation

$$\begin{aligned}\frac{1}{2\pi i}\int_X F(\varphi(x))\,\varphi'(x)\,dm(x) &= \tau([T_{F(\varphi)}^m, T_\varphi^m])\\ &= \tau([F(T_\varphi^m), T_\varphi^m]) = \tau_\varphi(F, z)\\ &= \frac{1}{\pi}\int_{|z| \leqslant \|\varphi\|_\infty} \bar{\partial}F(z)\, g(\varphi; z)\, d\mathfrak{L}^2(z)\\ &= \frac{1}{\pi}\int_{\mathbb{R}^2} (\bar{\partial}fF)(z)\, g(\varphi; z)\, d\mathfrak{L}^2(z).\end{aligned}$$

Since every polynomial in z and $\bar{z}$, restricted to the disc $|z| \leqslant \|\varphi\|_\infty$, can be written $\bar{\partial}(fF)$ for some polynomial F, it suffices to show that

$$\frac{1}{\pi}\int_{\mathbb{R}^2} \bar{\partial}(fF)(z)\, h(z)\, d\mathfrak{L}^2(z) = \frac{1}{2\pi i}\int_X F(\varphi(x))\,\varphi'(x)\,dm(x). \tag{25.1}$$

By Fubini's theorem, the left-hand integral in Eq. (25.1) equals

$$\frac{1}{2\pi i}\int_X \left(\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{-\bar{\partial}(fF)(z)}{\varphi(x)-z}\, d\mathfrak{L}^2(z)\right) \varphi'(x)\, dm(x).$$

On the other hand, by Green's theorem, we may write $(1/\pi)\int_{\mathbb{R}^2} (\bar{\partial}(fF)(z)/(\lambda - z))\, d\mathfrak{L}^2(z) = -(fF)(\lambda) + G(\lambda)$, where G is an entire function. Hence,

$$\begin{aligned}&\frac{1}{\pi}\int_{\mathbb{R}^2} \bar{\partial}(fF)(z)\, h(z)\, d\mathfrak{L}^2(z)\\ &\quad = \frac{1}{2\pi i}\int_X F(\varphi(x))\,\varphi'(x)\,dm(x) + \frac{1}{2\pi i}\int_X G(\varphi(x))\,\varphi'(x)\,dm(x).\end{aligned}$$

Since G is entire, G has a primitive, L, say, and so the last term is

$$\begin{aligned}&\frac{1}{2\pi i}\int_X \frac{d}{ds} L(\varphi(x+s))\Big|_{s=0}\, dm(x)\\ &\quad = \frac{1}{2\pi i}\frac{d}{ds}\left(\int_X L(\varphi(x+s))\, dm(x)\right)\Big|_{s=0}.\end{aligned}$$

Since m is invariant, the integral $\int_X L(\varphi(x+s))\, dm(x)$ is constant in s, and so its derivative is zero. This verifies Eq. (25.1) and completes the proof.

25.2. THEOREM. *If $(X, \mathbb{R})$ is minimal and if $\varphi \in C(X)$ is invertible, then the Breuer–Fredholm index of T^m_φ is $-\mu(\varphi; m)$.*

Proof. We give two closely related proofs. The first uses the Pincus principal function, the second avoids it.

For the first, note that since Index is continuous [Br], and since $C^1(X)$ is dense in $C(X)$, we may assume that $\varphi \in C^1(X)$. Then by [CaP1], the Pincus principal function $g(\varphi; \cdot)$ is constant on each connected component of the complement of $\sigma_{\mathrm{ess}}(T^m_\varphi)$, and if U is such a component, then for $z \in U$, $\mathrm{Index}(T^m_f - z) = g(\varphi; z)$. By Theorem 25.1, and Lemma 4.1, we see, then, that $\mathrm{Index}(T^m_\varphi) = g(\varphi; 0) = g(\varphi; 0) = (-1/2\pi i) \int_X (\varphi'(x)/\varphi(x))\, dm(x) = -\mu(\varphi; m)$.

For our alternate proof, we assume both that $\varphi \in C^1(X)$ and that φ is unimodular. We may do this by the homotopy invariance of Index [Br]. Denote the polar decomposition of T^m_φ by UA, where U is a partial isometry, $A = ((T^m_\varphi)^*(T^m_\varphi))^{1/2}$, and the initial space of U coincides with the initial space of T^m_φ. Then $\mathrm{Index}(T^m_\varphi) = \tau(I - U^*U) - \tau(I - UU^*) = \tau([U, U^*])$. But also,

$$\begin{aligned} U - T^m_\varphi &= U(I - A) = U(I - A^2)(I + A)^{-1} \\ &= U(T^m_{|\varphi|^2} - (T^m_\varphi)^*(T^m_\varphi))(I + A)^{-1} \in \mathfrak{K}_1(\mathfrak{N}) \end{aligned}$$

since $T^m_{|\varphi|^2} - (T^m_\varphi)^*(T^m_\varphi) \in \mathfrak{K}_1(\mathfrak{N})$. Consequently, by Theorem 23.1, $\mathrm{Index}(T^m_\varphi) = \tau([U, U^*]) = \tau([T^m_\varphi, (T^m_\varphi)^*]) = -(1/2\pi i) \int_x \varphi'(x)\, \overline{\varphi(x)}\, dm(x) = -\mu(\varphi; m)$.

25.3. *Remark.* This is a continuation of Remark 4.3. We follow the notation there. Let $\varphi \in C(X)^{-1}$ and view φ as an element of $K^1(X)$. Then $[\omega_\varphi]$ is the Chern class $\mathrm{ch}^*(\varphi)$ under the mapping $\mathrm{ch}^*: K^*(X) \to H^*(X)$. On the other hand, the exact sequence

$$0 \to \mathfrak{C}_m \to \mathfrak{T}_m \to C(X) \to 0$$

gives rise to an element α in $\mathrm{Ext}^{\mathfrak{N}}(X)$ (see [Fil]). Unfortunately, for II_∞-factors, one does not know the exact duality between $\mathrm{Ext}^{\mathfrak{N}}_*(X)$ and $K^*(X)$. Our calculation of $\mathrm{Index}(T^m_\varphi)$ suggests, however, that if $(X, \mathbb{R})$ is smooth and given by a vector field Z, and if [C] is the Ruelle–Sullivan current associated to Z and m, then [C] appears to be the Chern class of α under the dual of ch^*, ch_*. Assuming that $[C] = \mathrm{ch}_*(\alpha)$, we arrive at the following analogue of the usual index formula using Theorem 25.2 and Remark 4.3,

$$\operatorname{Index}(T_\varphi^m) = -\langle \mathrm{ch}_*(\alpha), \mathrm{ch}^*(\varphi) \rangle.$$

26. If we were dealing with classical Toeplitz operators with continuous symbols on the circle, then one could deduce the spectrum of the Toeplitz operator T_φ^m from Theorem 25.2. Indeed, one knows from Coburn's theorem [Cb] that if T_φ^m is Fredholm, then T_φ^m is invertible if and only if $\operatorname{Index}(T_\varphi^m) = 0$. Unfortunately, since Breuer–Fredholm operators need not have closed range, we are unable to decide if an analogue of Coburn's theorem holds in our setting. However, if one assumes that $(X, \mathbb{R})$ is strictly ergodic and that the mean motion μ, regarded as a homomorphism of $H^1(X, \mathbb{Z})$, is injective, then we can determine the spectrum of T_φ^m from Theorem 25.2.

26.1. THEOREM. *Suppose that $(X, \mathbb{R})$ is strictly ergodic and let m be the unique invariant probability measure on X. Suppose, too, that the mean motion μ determines an injective homomorphism of $H^1(X, \mathbb{Z})$. Then for $\varphi \in C(X)$, T_φ^m is invertible if and only if φ is invertible in $C(X)$ and $\mu(\varphi) = 0$.*

Proof. Suppose φ is invertible in $C(X)$ and $\mu(\varphi) = 0$. Then, by hypothesis, $\varphi = \exp(\psi)$ for some $\psi \in C(X)$. Since the flow is strictly ergodic, $A(X, \mathbb{R})$ is a Dirichlet algebra on X. Consequently, there are functions $a, b \in A(X, \mathbb{R})$ such that $\|\psi - (a + \bar{b})\|_\infty < \frac{1}{4}$. Writing $\varphi_1 = \exp(\bar{b})$, $\varphi_2 = \exp(\psi - (a + \bar{b}))$, and $\varphi_3 = \exp(a)$, we see that $T_\varphi^m = T_{\varphi_1}^m T_{\varphi_2}^m T_{\varphi_3}^m$. But $T_{\varphi_1}^m$ and $T_{\varphi_3}^m$ are invertible, with $(T_{\varphi_i}^m)^{-1} = T_{\varphi_i^{-1}}^m$, $i = 1, 2$, and $T_{\varphi_2}^m$ is invertible because $\|I - T_{\varphi_2}^m\| < 1$. Thus T_φ^m is invertible. Conversely, if T_φ^m is invertible, then T_φ^m is Fredholm with $\operatorname{Index}(T_\varphi^m) = 0$. By Theorems 24.4 and 25.2, φ is invertible in $C(X)$ and $\mu(\varphi) = 0$.

27. One bit of evidence that indicates that strict ergodicity may not be as necessary an hypothesis as our work might imply is the following consequence of Theorem 25.2.

27.1. THEOREM. *Suppose that $\varphi \in A(X, \mathbb{R})$ and that $(X, \mathbb{R})$ is minimal. If m is an invariant, ergodic probability measure on X, then T_φ^m is invertible if and only if φ is invertible in $C(X)$ and $\mu(\varphi; m) = 0$.*

Proof. Since $\varphi \in A(X, \mathbb{R})$, $T_\varphi^m \xi = \varphi \xi$ for all $\xi \in H^2(X \times \mathbb{R})$. Consequently, T_φ^m is bounded below if and only if φ is invertible in $C(X)$. But in this case, then, we see that T_φ^m is invertible if and only if $\operatorname{Index}(T_\varphi^m) = 0$

because $\operatorname{Index}(T_\varphi^m) = -\tau(N(T_\varphi^m)^*)$. The result now follows from Theorem 25.2.

27.2. COROLLARY. *If $(X, \mathbb{R})$ is minimal, and if $\varphi \in A(X, \mathbb{R})$, then φ is invertible in $A(X, \mathbb{R})$ if and only if φ does not vanish on X and $\mu(\varphi; m) = 0$, where m is any invariant, ergodic, probability measure.*

Proof. The map $\varphi \to T_\varphi^m$ is an isometric representation of $A(X, \mathbb{R})$ into $\mathfrak{N}$. So, φ is invertible in $A(X, \mathbb{R})$ if and only if T_φ^m is invertible. Now apply Theorem 27.1.

27.3. *Remark.* We do not know of a proof of Corollary 27.2 that avoids operator theory. If the flow were strictly ergodic, then from the form of the maximal ideal space of $A(X, \mathbb{R})$ (cf. Section 5.4) we could deduce Corollary 27.2 using the analysis of Part I.

28. Let $\varphi \in A(X, \mathbb{R})$ and $y > 0$. Set $\varphi_y = \varphi * P_{iy}$, where P_{iy} is the Poisson kernel for the upper half-space. (In Part I, we would have written $F(\cdot; iy)$ for φ_y; that notation is a bit cumbersome for us here.) Then, of course, $\varphi_y \in C^1(X)$ and so, if m is any invariant ergodic probability measure on X, we can compute the Pincus principal function of T_φ^m using Theorem 25.1, $g(\varphi_y; z) = (-1/2\pi i) \int \varphi_y'(x)/(\varphi_y(x) - z)\, dm(x)$ for $\mathfrak{L}^2$-almost all z. The exceptional null set depends on y. On the other hand, we showed in Corollary 16.3 that if $(X, \mathbb{R})$ is strictly ergodic, then there is a planar null set outside of which the mean motion $\mu(\varphi - z; x, y)$, defined in Section 8.1, exists and coincides with $(\partial/\partial y)\, \Phi(\varphi - z; y)$ independently of x. If z is fixed so that the integrand giving $g(\varphi_y; z)$ is in $L^1(m)$, then by the individual ergodic theorem there is a set $M_z \subseteq X$ with $m(M_z) = 0$ such that for $x \in X \backslash M_z$, $\varphi_y(x+t)/(\varphi_y(x+t) - z)$ is locally integrable on $\mathbb{R}$ and $\int_X \varphi_y'(x)/(\varphi_y(x) - z)\, dm(x) = \lim_{T \to \infty} (1/2T) \int_{-T}^T \varphi_y'(x+t)/(\varphi_y(x+t) - z)\, dt$. Since $\varphi_y'(x+t)/(\varphi_y(x+t) - z)$ is analytic in $z = t + iy$, its local integrability implies that $\varphi_y(x+t) - z \neq 0$ for all t and all $x \notin M_z$. Thus, in the notation of Section 4, the mean motion $\mu(\varphi_y - z, x)$ exists for all $x \notin M_z$ and satisfies

$$\begin{aligned}\mu(\varphi_y - z; x) &= \frac{1}{2\pi i} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \frac{\varphi_y'(x+t)}{\varphi_y(x+t) - z}\, dt \\ &= \frac{1}{2\pi i} \int_X \frac{\varphi_y'(x)}{\varphi_y(x) - z}\, dm(x).\end{aligned}$$

We therefore arrive at the following result which is an analogue of [CaP2, Theorem 6].

28.1. THEOREM. *Suppose that $(X, \mathbb{R})$ is strictly ergodic, that $\varphi \in A(X, \mathbb{R})$ is not identically zero, and that $y > 0$ is fixed. Then there is a planar null set*

E_y such that for $z \notin E_y$, the mean motion $\mu(\varphi - z; x - y)$ exists for all $x \in X$ and the following equation holds,

$$-g(\varphi_y; z) = \mu(\varphi - z; x, y) = \frac{\partial}{\partial y} \Phi(\varphi - z; y).$$

28.2. COROLLARY (cf. [CaP2, Theorem 7]). *With the hypotheses as in Theorem 28.1, for each pair $\{y_1, y_2\}$, with $0 < y_1 < y_2$, there is a planar null set $E(y_1, y_2)$ such that for $z \notin E(y_1, y_2)$*

$$H(\varphi - z; x; y_1, y_2) = g(\varphi_{y_2}; z) - g(\varphi_{y_1}; z).$$

Here, as in Section 16, $H(\varphi - z; x; y_1, y_2)$ is the relative frequency of zeros of $F(x; \cdot) - z$ in the strip $y_1 < \operatorname{Im} \zeta < y_2$, where F is the function associated with φ in Section 5.2. Of course Corollary 28.2 is just a combination of Theorem 28.1 and Corollary 16.4. Thus nothing more in the way of a proof is necessary.

If $\varphi \in A(X, \mathbb{R})$ does not vanish on X, then T^m_φ is a Breuer–Fredholm operator in $\mathfrak{N}$ and has index $-\mu(\varphi; m)$ by Theorem 25.2. On the other hand, for $y > 0$ sufficiently small, φ_{y_1} is close to φ and so

$$\begin{aligned} \operatorname{Index}(T^m_\varphi) &= \operatorname{Index}(T^m_{\varphi_{y_1}}) \\ &= g(\varphi_{y_1}; 0) = g(\varphi_{y_2}; 0) - H(\varphi; x;, y_1, y_2) \end{aligned} \tag{28.1}$$

for any $x \in X$, where y_1 and y_2 are points such that $(\partial/\partial y)\, \Phi(\varphi; \cdot)$ exists and coincides with $-g(\varphi_{y_i}; 0)$, $i = 1, 2$. Here we are assuming that $(X, \mathbb{R})$ is strictly ergodic so that we can invoke Jensen's formula, formula (14.4). We would like to take the limit as $y_2 \to \infty$, and assert that $\operatorname{Index}(T^m_\varphi)$ is the negative of the density of some zeros of φ in the half-plane erected over any orbit in X. There are two problems with this: (1) We do not know if $\lim_{y \to \infty} g(\varphi_y; 0)$ exists. This is related to a famous problem of Jessen and Tornehave [JT, p. 192] and is solved in the almost periodic case by Levin (cf. [L, Chap. VI, Theorem 6] for his solution and references). (2) When the limit exists, it may not be zero. In fact, in cases we can decide, when the limit exists it is precisely the negative of the infimum of the spectrum, in the sense of spectral synthesis, of φ. This, in turn, may be viewed as the order of the zero that the Gelfand transform of φ has at the center of the maximal ideal space of $A(X, \mathbb{R})$, even though the order may not be an integer. Thus, it looks promising that after adding in that zero, $\operatorname{Index}(T^m_\varphi)$ is the ("average") number of zeros of the Gelfand transform of φ in the maximal ideal space of $A(X, \mathbb{R})$.

Here is a precise statement of what happens when the Gelfand transform of a function does not vanish in center of the maximal ideal space of $A(X, \mathbb{R})$.

28.3. COROLLARY. *Suppose that* $(X, \mathbb{R})$ *is strictly ergodic, that* $\varphi \in A(X, \mathbb{R})$ *does not vanish on* X, *and that* $\int_X \varphi \, dm \neq 0$. *Then there exist* y_1 *and* y_2 *with* $0 < y_1 < y_2 < \infty$ *so that the zeros of the function* $F(x; z)$ *associated with* φ *in Section* 5.2 *are located in the strip* $y_1 < \operatorname{Im} z < y_2$ *for every* $x \in X$ *and, moreover,* Index $T^m_\varphi = -H(\varphi; x; y_1, y_2)$ *for all* $x \in X$.

Proof. As noted above, since $\varphi_y \to \varphi$ as $y \to 0$, there is a y_1 such that φ_y is nonvanishing for all $y \leqslant y_1$. Consequently, the zeros of $F(x; \cdot)$ lie above $\operatorname{Im} z = y_1$. On the other hand, $\lim_{y \to \infty} \varphi_y(x) = \int \varphi \, dm$ uniformly in x, as may be seen from the discussion in Section 5.4. Thus, there is a y_2 such that for $y \geqslant y_2$ not only is φ_y zero-free on X, but $T^m_{\varphi_y}$ is invertible. Consequently, the zeros of $F(x; \cdot)$ lie below y_2 for all x and Index $T^m_\varphi = -H(\varphi; x; y_1, y_2)$ because $g(\varphi_{y_2}; 0) = 0$.

29. In this section we develop some analogues of the continuity results in [CaP2]. We say that a sequence $\{\varphi_n\}_{n=1}^\infty$ is *bounded* in $C^1(X)$ if $\sup_n \|\varphi_n\|$ and $\sup_n \|\varphi'_n\|$ are finite.

29.1. LEMMA. *Suppose that* $(X, \mathbb{R})$ *is minimal and that* m *is an invariant, ergodic, probability measure on* X.

(i) *For each* $\varphi \in C^1(X)$, *the Pincus principal function* $g(\varphi; \cdot)$ *for* T^m_φ *belongs to* $L^{1+\varepsilon}(\mathbb{R}^2, \mathfrak{L}^2)$ *for each* ε, $0 \leqslant \varepsilon < 1$.

(ii) *If* $\{\varphi_n\}_{n=1}^\infty$ *is a bounded sequence in* $C^1(X)$ *such that* $\lim_{n \to \infty} \varphi_n = \varphi$ *a.e. and* $\lim_{n \to \infty} \varphi'_n = \varphi'$ *a.e. for some* $\varphi \in C^1(X)$, *then* $\lim_{n \to \infty} g(\varphi_n; \cdot) = g(\varphi; \cdot)$ *in the norm of* $L^{1+\varepsilon}(\mathbb{R}^2, \mathfrak{L}^2)$, $0 \leqslant \varepsilon < 1$.

Proof. (i) is an immediate consequence of Theorem 25.1 and Fubini's theorem. For (ii), fix ε, $0 \leqslant \varepsilon \leqslant 1$, and observe that since

$$g(\varphi_n; z) = \frac{1}{2\pi i} \int_X \frac{\varphi'_n(x)}{\varphi_n(x) - z} \, dm(x),$$

while $\{\varphi_n\}_{n=1}^\infty$ is bounded in $C^1(X)$, the sequence $\{|\varphi'_n(x)/(\varphi_n(x) - z)|^{1+\varepsilon}\}_{n=1}^\infty$ is uniformly integrable with respect to $m \times \mathfrak{L}^2$; i.e., for each $\sigma > 0$ there is a $\delta > 0$ so that

$$\int_\Delta \int \left| \frac{\varphi'_n(x)}{\varphi_n(x) - z} \right|^{1+\varepsilon} dm \, d\mathfrak{L}^2 < \delta$$

whenever $m \times \mathfrak{L}^2(\Delta) < \sigma$. Our hypothesis implies that $\varphi'_n/\varphi_n - z \to \varphi'/\varphi - z$ a.e. $(m \times \mathfrak{L}^2)$ and so a version of the dominated convergence theorem implies that $\varphi'_n/\varphi_n - z \to \varphi'/\varphi - z$ in $L^{1+\varepsilon}(X \times \mathbb{R}^2, m \times \mathfrak{L}^2)$. Fubini's theorem, then, implies that $g(\varphi_n; \cdot) \to g(\varphi; \cdot)$ in $L^{1+\varepsilon}(\mathbb{R}^2, \mathfrak{L}^2)$.

The following is our analogue of [CaP2, Theorem 1]. Recall that in Corollay 16.3 we showed that if $\varphi \in A(X, \mathbb{R})$ ($(X, \mathbb{R})$ strictly ergodic) and if $y>0$, then outside a planar null set of z, $\mu(\varphi - z; x, y) = (\partial/\partial y)\, \Phi(\varphi - z; y)$ for all $x \in X$. Consequently, we shall omit all reference of x and simply write $\mu(\varphi - z; y)$ for $\mu(\varphi - z; x, y)$.

29.2. THEOREM. *Suppose that $(X, \mathbb{R})$ is strictly ergodic, that m is the unique invariant probability measure on X, and that $y > 0$ is fixed.*

(i) *If $\varphi \in A(X, \mathbb{R})$ and if $0 \leqslant \varepsilon < 1$, then $\mu(\varphi - z; y)$, as a function of z, lies in $L^{1+\varepsilon}(\mathbb{R}^2, \mathfrak{L}^2)$.*

(ii) *If $\{\varphi_n\}_{n=1}^{\infty}$ is a sequence in $A(X, \mathbb{R})$ such that $\sup_n \|\varphi_n\| < \infty$ and that $\{\varphi_n\}_{n=1}^{\infty}$ converges almost everywhere with respect to m to a function φ in $A(X, \mathbb{R})$, then for each ε, $0 \leqslant \varepsilon < 1$, the functions of z, $\mu(\varphi_n - z; y)$, converge to $\mu(\varphi - z; y)$ in $L^{1+\varepsilon}(\mathbb{R}^2; \mathfrak{L}^2)$.*

Proof. Assertion (1) is an immediate consequence of Theorem 28.1 and Lemma 29.1(i). Assertion (ii) will follow from Theorem 28.1 and Lemma 29.1(ii) once it is noted that our hypotheses imply that $\sup_n \|(\varphi_n')_y\|_\infty < \infty$ and $\lim_{n \to \infty} (\varphi_n)_y' = \varphi_y'$ a.e. m. But this is a straightforward calculation based on the observation that $\varphi_y = \varphi * P_{iy}$ for all $F \in A(X, \mathbb{R})$.

30. In this section we investigate the consequences of the assumption that our flow has vanishing first cohomology. We formulate our results under the more general hypothesis that the mean motions determined by the quasi-regular points, viewed as a real-valued homomorphism of $H^1(X, \mathbb{Z})$, have zero range (cf. Section 4).

We note that there are strictly ergodic flows with $H^1(X, \mathbb{Z}) = 0$. Thus our hypotheses will not be vacuous. For example, it is known [FH] that if a smooth manifold admits a free, smooth action of the 2-torus, then the manifold admits a smooth strictly ergodic flow. On the sphere S^{2p+1}, the vector field

$$(x_0, y_0, x_1, y_1, ..., x_p, y_p) \to (-y_0, x_0, -y_1, x_1, ..., -y_p, x_p)$$

gives rise to a free C^∞ action of the 1-torus. Consequently, $X = S^{2p+1} \times S^{2q+1}$ carries a free C^∞ action of the 2-torus. So, if $p, q \geqslant 1$, we obtain a strictly ergodic flow on X with $H^1(X, \mathbb{Z}) = 0$. The following is an immediate consequence of Corollary 27.2.

30.1. THEOREM. *Suppose that $(X, \mathbb{R})$ is minimal and suppose that each quasi-regular point x in the support of some invariant ergodic probability* *measure induces the zero homomorphism on $H^1(X, \mathbb{Z})$. Then for all $\varphi \in A(X, \mathbb{R})$, the range of φ on X, $\varphi(X)$, coincides with the range of the Gelfand transform of φ on the maximal ideal space of $A(X, \mathbb{R})$.*

30.2. COROLLARY. *With the hypotheses as above, if φ is a nonconstant function in $A(X, \mathbb{R})$, then $\varphi(X)$ contains an open set in $\mathbb{C}$. In particular, there are no inner functions in $A(X, \mathbb{R})$; i.e., if $\varphi \in A(C, \mathbb{R})$ and if $|\varphi(x)| = 1$ for all $x \in X$, then φ is constant.*

Proof. It is shown in [M4] that even if $(X, \mathbb{R})$ is not strictly ergodic (but is minimal, say), then the maximal ideal space of $A(X, \mathbb{R})$ contains an open set homeomorphic to $X \times [0, \infty)$; the Gelfand transform, then, of φ, evaluated at (x, y) is given by $F(x; iy)$ where F is the function associated with F in Section 5.2. Since, for x fixed, $F(x+t; iy) = F(x; t+iy)$ is an analytic function of $t+iy$, the corollary follows from the open mapping theorem for analytic functions.

30.3. *Remarks.* (i) If $(X, \mathbb{R})$ is strictly ergodic and if $H^1(X, \mathbb{Z}) = 0$, then $A(X, \mathbb{R})$ is a Dirichlet algebra with the property that for each $\varphi \in A(X, \mathbb{R})$ the range of φ on X, $\varphi(X)$, coincides with the closure of the range of the Gelfand transform of φ restricted to any nontrivial Gleason part. It strikes us as remarkable that Dirichlet algebras with this property exist. Looking at things a little differently, observe that if $\varphi \in A(X, \mathbb{R})$ and if F is the function associated with φ in Section 5.2, then for each $x \in X$, the function $F(x; \cdot)$ has the property that $\{F(x; t) \mid t \in \mathbb{R}\}^{\mathrm{cl}} = \{F(x; z) \mid \operatorname{Im} z > 0\}^{\mathrm{cl}}$. Thus the curve $t \to F(x; t)$ oscillates wildly at infinity. Note, too, that functions φ may be chosen so that $F(x; t)$ is as smooth in t as one might like. In particular, if $\psi \in C(X)$ and if $f \in L^1(\mathbb{R})$ are chosen such that $\varphi \equiv \psi * f \neq 0$ while $\hat{f}$ has compact support in $[0, \infty)$ (such choice is always possible), then $F(x; z)$ will be entire in z. Now it is possible to construct such functions directly. Indeed, Boris Mityagin pointed out to us that the function $\mathrm{f}(z) = \cos(2\pi\beta z) \cdot e^{iz}$, where β is irrational and $|\beta| < 1$, is such a function. However, what is surprising is that functions with this property—even whole algebras of functions with this property—exist in such profusion.

(ii) It is surprising, too, that $A(X, \mathbb{R})$ has no nonconstant inner functions under the hypotheses of Theorem 30.1. The reason is that the weak-$*$ closure of $A(X, \mathbb{R})$ in $L^\infty(m)$, $H^\infty(m)$, is a weak-$*$ Dirichlet algebra [M1] that always contains lots of inner functions by a theorem of Douglas and Rudin [DR]. In fact, they show that the space of quotients of inner functions is uniformly dense in the unimodular functions in $L^\infty(m)$.

Expanding on Corollary 30.2, we have

30.4. THEOREM. *Suppose that $(X, \mathbb{R})$ is minimal and that the homomorphism of $H^1(X, \mathbb{Z})$ induced by the mean motion is zero for each point in the support of some invariant ergodic probability measure m. Then $\mathfrak{T}_m$ contains no nonunitary isometries.*

Proof. Let V be an isometry in $\mathfrak{T}_m$ and write $V = T^m_\varphi + K$, where $\varphi \in C(X)$ and $K \in \mathfrak{C}_m$ (cf. Section 24). The equation $V^*V = I$ implies that $T^m_{|\varphi|^2} = (T^m_\varphi)^*(T_\varphi) + K_1 = V^*V + K_1 + K_2 = I + K_1 + K_2$, for some $K_1, K_2 \in \mathfrak{C}_m$. Consequently, the essential spectrum of $T^m_{|\varphi|^2}$ is $\{1\}$; so $|\varphi| = 1$ on X, by Corollary 24.5. Thus T_φ is Fredholm, again by Corollary 24.5, and the index of T_φ is zero by hypothesis and Theorem 25.2. Thus $\text{Index}(V) = 0$. Since V has zero kernel and closed range, we conclude from this that V is unitary.

30.5. *Remark.* This theorem naturally suggests the question: Are there partial isometries in $\mathfrak{T}_m$? And further: Are there nontrivial projections in $\mathfrak{T}_m$ when $H^1(X, \mathbb{Z}) = \{0\}$? Since every element of $\mathfrak{T}_m$ has the form $T^m_\varphi + K$, for some $K \in \mathfrak{C}_m$, a projection in $\mathfrak{T}_m$ is either finite or cofinite dimensional. So the real question is: When $H^1(X, \mathbb{Z}) = \{0\}$, are there nontrivial projections in $\mathfrak{C}_m$? It is known that the C^*-algebra $C(X) \rtimes \mathbb{R}$, contains no projections when $H^1(X, \mathbb{Z}) = \{0\}$ [Co3], so since $\mathfrak{C}_m$ is not too far removed from $C(X) \rtimes \mathbb{R}$, one would guess that the answer to this question is no.

31. It is natural to try to determine the extent to which the C^*-algebras $\mathfrak{T}_x$ and $\mathfrak{T}_m$ determine the flow $(X, \mathbb{R})$. In particular, it is natural to ask for a description of the automorphism group of $\mathfrak{T}_m$. It appears that these are difficult questions and all we can do at the present is make the following modest contribution. First we give a lemma.

31.1. LEMMA. *Suppose that $\varphi \in L^\infty(\mathbb{R})$ and that on $H^2(\mathbb{R})$, the equation $T_\varphi - T_\varphi = T_{|\varphi|^2}$ holds. Then $\varphi \in H^\infty(\mathbb{R})$.*

Proof. By a conformal change of variables we may assume that $\varphi \in L^\infty(\mathbb{T})$ and that $T_{\bar\varphi} T_\varphi = T_{|\varphi|^2}$ on $H^2(\mathbb{T})$. If P is the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$, then we have $\|P\varphi\|^2 = \|T_\varphi 1\|^2 = (T_{\bar\varphi} T_\varphi 1, 1) = (T_{|\varphi|^2} 1, 1) = \int_{\mathbb{T}} |\varphi(z)|^2 \, dm(z) = \|\varphi\|^2$. Then $P\varphi = \varphi$ and $\varphi \in L^\infty(\mathbb{T}) \cap H^2(\mathbb{T}) = H^\infty(\mathbb{T})$.

31.2. THEOREM. *For $i = 1, 2$, let $(X_i, \mathbb{R})$ be a strictly ergodic flow and let m_i be the unique invariant probability measure on X_i. If $\rho: \mathfrak{T}_{m_i} \to \mathfrak{T}_{m_2}$ is a C^*-isomorphism such that $\rho(T^{m_1}_\varphi)$ is a Toeplitz operator in $\mathfrak{T}_{m_2}$ for each $\varphi \in C(X_1)$, then there is a homeomorphism $\pi: X_1 \to X_2$ and $\lambda > 0$ so that*

(i) $\rho(T_\varphi^{m_1}) = T_{\varphi\circ\pi^{-1}}^{m_2}$ *for* $\varphi \in C(X_1)$, *and*

(ii) $\pi(x+t) = \pi(x) + \lambda t$, *for all* $x \in X$, $t \in \mathbb{R}$.

Conversely, if π *is a homeomorphism from* X_1 *onto* X_2 *satisfying* (ii) *for some* λ, *then* ρ, *defined by* (i) *extends to a* C^**-isomorphism from* $\mathfrak{T}_{m_1}$ *onto* $\mathfrak{T}_{m_2}$.

Proof. Let $\rho: \mathfrak{T}_{m_1} \to \mathfrak{T}_{m_2}$ be an isomorphism such that $\rho(T_\varphi^{m_1})$ is a Toeplitz operator in $\mathfrak{T}_{m_2}$ for each $\varphi \in \mathfrak{T}_{m_1}$. Since $\mathfrak{C}_{m_i}$ coincides with the commutator ideal in $\mathfrak{T}_{m_i}$ by Lemma 24.1, we conclude that $\rho(\mathfrak{C}_{m_1}) = \mathfrak{C}_{m_2}$. But, then, passing to the quotients we see that ρ induces an isomorphism from $C(X_1) \cong \mathfrak{T}_{m_1}/\mathfrak{C}_{m_1}$ onto $C(X_2) \cong \mathfrak{T}_{m_2}/\mathfrak{C}_{m_2}$. It follows, then, that there is a homeomorphism $\pi: X_1 \to X_2$ such that the isomorphism between $C(X_1)$ and $C(X_2)$ induced by ρ is implemented by π. At the level of Toeplitz operators, then, we conclude that for each $\varphi \in C(X_1)$ there is a $K_\varphi \in \mathfrak{C}_{m_2}$ such that $\rho(T_\varphi^{m_1}) = T_{\varphi\circ\pi^{-1}}^{m_2} + K_\varphi$. However, by hypothesis, $\rho(T_\varphi^{m_1})$ is a Toeplitz operator, so $K_\varphi = 0$ by Lemma 24.3. Thus (i) is satisfied. To see that (ii) is satisfied, simply note that it suffices to show $\varphi\circ\pi^{-1} \in A(X_2, \mathbb{R})$ for each $\varphi \in A(X_1, \mathbb{R})$ by the main theorem in [M4]. To do that, we employ Lemma 31.1 as follows. If $\varphi \in A(X_1, \mathbb{R})$, then $T_{\bar\varphi}^{m_1} T_\varphi^{m_1} = T_{|\varphi|^2}^{m_1}$. But, then, also, $T_{\bar\varphi\circ\pi^{-1}}^{m_2} T_{\varphi\circ\pi^{-1}}^{m_2} = \rho(T_{\bar\varphi}^{m_1} T_\varphi^{m_1}) = \rho(T_{|\varphi|^2}^{m_1}) = T_{|\varphi\circ\pi^{-1}|^2}^{m_2}$. Hence for each $x \in X_2$ the Toeplitz operator $T_{\varphi\circ\pi^{-1}}^x$ on $H^2(\mathbb{R})$ satisfies the equation $T_{\bar\varphi\circ\pi^{-1}}^x T_{\varphi\circ\pi^{-1}}^x = T_{|\varphi\circ\pi^{-1}|^2}^x$. By Lemma 31.1, the restriction of $\varphi\circ\pi^{-1}$ to the orbit through x is in $H^\infty(\mathbb{R})$. Thus, since $x \in X_2$, is arbitrary, we conclude that $\varphi\circ\pi^{-1} \in A(X_2, \mathbb{R})$.

32. In this last section, we consider the Fredholm theory of systems of Toeplitz operators. Our results generalize those of Schaeffer [Sch], who considered Toeplitz operators on certain almost periodic flows. On page 492 of [CaP2], Carry and Pincus note that in this setting calculations involving the principal function can be used to prove Schaeffer's index theorem in a very straightforward manner. Since most of the ground work has been laid, we will be brief and simply cite the necessary changes needed in earlier arguments to assemble our index formula.

Throughout, $(X, \mathbb{R})$ will be a minimal flow and m will be an invariant ergodic probability measure. We form the Toeplitz algebras $\mathfrak{T}_x$, $x \in X$, and $\mathfrak{T}_m$ described above as well as the II_∞ factor $\mathfrak{N}$. Fix a positive interger n and tensor each algebra $\mathfrak{T}_x$, $\mathfrak{T}_m$, and $\mathfrak{N}$ with the complex $n \times n$ matrices M_n. Elements in $\mathfrak{N} \otimes M_n$ may be viewed as $n \times n$ matrices over $\mathfrak{N}$, and so $\mathfrak{N} \otimes M_n$ is a II_∞ factor with trace τ_n defined by the formula

$$\tau_n((A_{ij})) = \sum_{i=1}^{n} \tau(A_{ii}),$$

where τ is the trace on $\mathfrak{N}$. The representations σ^x and σ^m of $C(X)$ give rise naturally to representations of $C(X)\otimes M_n$ on the Hilbert spaces $L^2(\mathbb{R})\otimes\mathbb{C}^n$ and $L^2(X\times\mathbb{R})\otimes\mathbb{C}^n$, respectively. We keep the same notation for the extensions. Thus, if $\Phi=(\varphi_{ij})\in C(X)\otimes M_n$ and if we view $L^2(\mathbb{R})\otimes\mathbb{C}^n$ as n-tuples of functions in $L^2(\mathbb{R})$, then $(\sigma^x(\Phi)\xi)_k=\sum_{j=1}^n \sigma^x(\varphi_{kj})\,\xi_j$. A similar formula holds for σ^m. For $\Phi=(\varphi_{ij})\in C(X)\otimes M_n$, then T^x_Φ is defined on $H^2(\mathbb{R})\otimes\mathbb{C}^n$ by the formula

$$T^x_\Phi=(P\otimes I_n)\,\sigma^x(\Phi)\,|\,H^2(\mathbb{R})\otimes\mathbb{C}^n.$$

Similarly, T^m_Φ is defined to be

$$(P\otimes I_n)\,\sigma^m(\Phi)\,|\,H^2(X\times\mathbb{R})\otimes\mathbb{C}^n.$$

The C^*-algebra generated by $\{T^x_\Phi\,|\,\Phi\in C(X)\otimes M_n\}$ is (naturally isomorphic to) $\mathfrak{T}_x\otimes M_n$ and likewise, $\{T^m_\Phi\,|\,\Phi\in C(X)\otimes M_n\}$ generates $\mathfrak{T}_m\otimes M_n$. Using Theorem 19, we see at once that $\mathfrak{T}_x\otimes M_n$ and $\mathfrak{T}_m\otimes M_n$ are isomorphic for every x. In fact, if i_n is the identity map on M_n and if ρ_x is the map defined in Theorem 19, then for $\Phi\in C(X)\otimes M_n$, $(\rho_x\otimes i_n)(T^x_\Phi)=T^m_\Phi$ and $\rho_x\otimes i_n$ extends to an isometry from $\mathfrak{T}_x\otimes M_n$ onto $\mathfrak{T}_m\otimes M_n$.

Lemma 22.1 is easily generalized to

32.1. Lemma. *If* $\Phi=(\varphi_{ij})\in C^1(X)\otimes M_n$, *then the commutator* $[H\otimes I_n,\sigma(\Phi)]$ *lies in* $\mathfrak{K}_2((L^\infty(m)\rtimes\mathbb{R})\otimes M_n)=\mathfrak{K}_2(L^\infty(m)\rtimes\mathbb{R})\otimes M_n$, *and*

$$\begin{aligned}\tau_n([H\otimes I_n,\sigma(\Phi)]^*[H\otimes I_n,\sigma(\Phi)])&=\frac{1}{\pi^2}\int_X\int_{\mathbb{R}}\sum_{i,j=1}^n\left|\frac{1}{s}\,(\varphi_{ij}(x+s)-\varphi_{ij}(x))\right|^2 ds\,dm(x)\\&=\frac{1}{\pi^2}\int_X\int_{\mathbb{R}}\left\|\frac{1}{s}\,(\Phi(x+s)-\Phi(x))\right\|_2^2 ds\,dm(x).\end{aligned}$$

Here, $\|\cdot\|_2$ *denotes the Hilbert–Schmidt norm on* M_n, *and, as in Lemma* 22.1, *we have omitted the superscript* m *on* H *and* σ *to lighten the notation.*

With Lemma 32.1 in hand, only minor changes in the proof of Theorem 23.1 and in the first part of the proof of Lemma 24.3 are necessary to prove

32.2. Lemma. *If* $\Phi=(\varphi_{ij})$ *and* $\Psi=(\psi_{ij})$ *are in* $C^1(X)\otimes M_n$ *and if* $\Phi(x)$

and $\Psi(x)$ *commute for all* x, *then both* $[T^m_\Phi, T^m_\Psi]$ *and* $T^m_\Phi T^m_\Psi - T^m_{\Phi\Psi}$ *lie in* $\mathfrak{K}_1(\mathfrak{K}\otimes M_n)$ *and*

$$\begin{aligned}\tau_n([T^m_\Phi, T^m_\Psi]) &= \sum_{i,j=1}^{n} \tau([T^m_{\varphi_{ij}}, T^m_{\psi_{ji}}])\\ &= \frac{-1}{2\pi i}\int_X \sum_{i,j=1}^{n} \varphi'_{ij}(x)\,\psi_{ji}(x)\,dm(x)\\ &= \frac{-1}{2\pi i}\int_X \operatorname{tr}(\Phi'(x)\,\Psi(x))\,dm(x). \qquad (32.1)\end{aligned}$$

Proof. We omit the superscript m to lighten the notation. Note that something like the hypothesis that $[\Phi(x), \Psi(x)]=0$ is necessary, since if Φ and Ψ are constant functions with nonzero commutator C then $[T_\Phi, T_\Psi]=I\otimes C$, which is not in $\mathfrak{K}_1(\mathfrak{N})$. Since $P\sigma(\Phi)Q\in\mathfrak{K}_1(\mathfrak{N}\otimes M_n)$ by Lemma 32.1, the first part of the proof of 24.3 shows that $T_\Phi T_\Psi - T_{\Phi\Psi}$ belongs to $\mathfrak{K}_1(\mathfrak{N}\otimes M_n)$. (This does not use the hypothesis that $[\Phi, \Psi]=0$.) But then, $[T_\Phi, T_\Psi]=(T_\Phi T_\Psi - T_{\Phi\Psi})+(T_{\Psi\Phi}-T_\Phi T_\Psi)$, provided $[\Phi, \Psi]=0$, and so $[T_\Phi, T_\Psi]$ belongs to $\mathfrak{K}_1(\mathfrak{N})$, too. To calculate $\tau_n[T_\Phi, T_\Psi]$, note that if $[\Phi, \Psi]=0$, then

$$\begin{aligned}&T_\Phi T_\Psi - T_\Psi T_\Phi\\ &\quad= P\sigma(\Phi)\,P\sigma(\Psi)P - P\sigma(\Psi)\,P\sigma(\Phi)P\\ &\quad= \tfrac{1}{2}[P\sigma(\Phi)\,H\sigma(\Psi)P - P\sigma(\Psi)\,H\sigma(\Phi)P\\ &\qquad+ \tfrac{1}{2}[P\sigma(\Phi)\,\sigma(\Psi)P - P\sigma(\Psi)\,\sigma(\Phi)P]\\ &\quad= -\tfrac{1}{2}[P\sigma(\Psi)\,H\sigma(\Phi)P - P\sigma(\Phi)\,H\sigma(\Psi)P]\\ &\quad= \tfrac{1}{2}P[\sigma(\Psi), \sigma(\Psi)]P - \tfrac{1}{2}[P\sigma(\Psi)\,H\sigma(\Phi)P - P\sigma(\Phi)\,H\sigma(\Psi)P]\\ &\quad= P[\sigma(\Phi)\,Q\sigma(\Psi) - \sigma(\Phi)\,Q\sigma(\Psi)]P.\end{aligned}$$

(Since $P\sigma(\Phi)Q\in\mathfrak{K}_2(\mathfrak{N}\otimes M_n)$, this also shows that $[T_\Phi, T_\Psi]\in\mathfrak{K}_1(\mathfrak{N}\otimes M_n)$.) This last expression is now easily seen to be

$$\tfrac{1}{4}[P(\sigma(\Phi)\,H\sigma(\Psi) - \sigma(\Psi)\,H\sigma(\Phi))P + Q(\sigma(\Phi)\,H\sigma(\Psi) - \sigma(\Psi)\,H\sigma(\Phi)Q]$$

whose trace in $\mathfrak{M}\otimes M_n$ sums to be

$$\frac{1}{\pi i}\int_X \operatorname{tr}(\Psi'(x)\,\Phi(x) - \Psi(x)\,\Phi'(x))\,dm(x).$$

If we now invoke the ergodic theorem and integrate by parts, as in Theorem 23.1 we arrive at Eq. (32.1).

32.2.1. COROLLARY. *For all* $\Phi, \Psi \in C(X) \otimes M_n$, $T^m_\Phi T^m_\Psi - T^m_{\Phi\Psi}$ *lies in* $\mathfrak{K}_\infty(\mathfrak{N}) \otimes M_n$.

The ideal in $\mathfrak{T}_m \otimes M_n$ generated by $\{T^m_\Phi T^m_\Psi - T^m_{\Phi\Psi} \mid \Phi, \Psi \in C(X) \otimes M_n\}$ is simply $\mathfrak{C}_m \otimes M_n$ and, likewise, the ideal in $\mathfrak{T}_x \otimes M_n$ generated by $\{T^x_\Phi T^x_\varphi - T^x_{\Phi\Psi} \mid \Phi, \Psi \in C(X) \otimes M_n\}$ is $\mathfrak{C}_x \otimes M_n$. The following lemma is immediate from Lemma 24.2.

32.3. LEMMA. *For each* x, $\mathfrak{T}_x \otimes M_n / \mathfrak{C}_x \otimes M_n$ *and* $\mathfrak{T}_m \otimes M_n / \mathfrak{C}_m \otimes M_n$ *are isomorphic and each is isomorphic to* $C(X) \otimes M_n$.

Since $\mathfrak{K}_\infty(\mathfrak{N} \otimes M_n) = \mathfrak{K}_\infty(\mathfrak{N}) \otimes M_n$; i.e., since a matrix is compact in $\mathfrak{N} \otimes M_n$ if and only if each entry is compact in $\mathfrak{N}$, Lemma 24.3 yields

32.4. LEMMA. *For each* $\Phi \in C(X) \otimes M_n$, T^m_Φ *lies in* $\mathfrak{K}_\infty(\mathfrak{N} \otimes M_n)$ *if and only if* $\Phi = 0$.

Recall that a matrix function Φ in $C(X) \otimes M_n$ is invertible in $C(X) \otimes M_n$ precisely when $\det(\Phi)$ never vanishes on X. Combining Lemmas 32.3 and 32.4, the first half of our index theorem for systems is proved just as for single operators, Theorem 24.4.

32.5. THEOREM. *Assume that* $(X, \mathbb{R})$ *is minimal and that* m *is an invariant, ergodic, probability measure on* X. *Then for each* $\Phi \in C(X) \otimes M_n$, T^m_Φ *is Fredholm in* $\mathfrak{N} \otimes M_n$ *if and only if* $\det(\Phi)$ *is invertible in* $C(X)$. *In this case, the Breuer–Fredholm index of* T^m_Φ *is* $-\mu(\det(\Phi); m)$.

Proof. We attend only to the index formula. Since the Breuer–Fredholm index is homotopy invariant [Br], as is $\mu(\cdot\,; m)$, we may assume without loss of generality that Φ is unitary-valued with entries in $C^1(X)$. Then, just as in the second proof of Theorem 25.2, we find that $\operatorname{Index}(T^m_\Phi) = \tau_n([T^m_\Phi, (T^m_\Phi)^*])$. By Lemma 32.2 and the fact that $[\Phi, \Phi^*] = 0$, this trace is

$$-\frac{1}{2\pi i} \int_X \operatorname{tr}(\Phi'(x)\, \Phi(x)^*)\, dm(x)$$

which can be calculated, via the ergodic theorem, as

$$-\frac{1}{2\pi i} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \operatorname{tr}(\Phi'(x+t)\, \Phi(x+t)^*)\, dt,$$

for all but a null set of x. Since Φ is unitary-valued, the integral in the limit is

$$\begin{aligned}\int_{-T}^{T} \frac{d}{dt}\log(\det(\Phi(x+t)))\,dt &= \log(\det(\Phi(x+T))) - \log(\det(\Phi(x-T)))\\ &= i\arg(\det(\Phi(x+T))) - i\arg(\det(\Phi(x-T)))\end{aligned}$$

for any continuous branch of $\arg(\det(\Phi(x+t)))$. This the limit is $-\mu(\det(\Phi); m)$, as promised.

Note added in proof. Ian Putnam and the second two authors of this paper have shown that the intrinsic C^*-algebra associated with a strictly ergodic flow, $\mathfrak{T}(X, \mathbb{R})$, discussed in Section 20, is indeed the appropriate universal object to be called *the* C^*-algebra of the flow. All the natural candidates for the C^*-algebra of Toeplitz operators on a flow, including $\mathfrak{T}_x$ and $\mathfrak{T}_m$, are naturally isomorphic $\mathfrak{T}(X, \mathbb{R})$. They have also calculated the K-theory of $\mathfrak{T}(X, \mathbb{R})$ and its commutator $\mathfrak{C}(X, \mathbb{R})$.

Lemma 24.1 is valid under the weaker hypothesis that $\mathbb{R}$ is freely acting on X; no assumption of strict ergodicity is necessary.

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